Classical Mechanics Problems: Problems Only

Justin Kulp

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Abstract

This is a series of problems developed for use in the new Classical Mechanics I and II courses at Lakehead University. The development of the courses involved a major content overhaul as the Lagrangian and Hamiltonian formalism took the drivers seat (the previous renditions focused on a force based approach to classical physics.) These problems were written so students could develop experience with numerical methods (in particular, experience with Mathematica) and get a feeling for how quickly problems can become non-analytic.

This is the "problems only" version of the project, people wishing to use any of the problems for their own coursework or study are free to do so. For my own interest I would love to be contacted to know what they're being used for. Emails containing complaints and errors are also encouraged.

1 Year 2: Coupled Oscillators

This question will give an introduction to Mathematica and its commands, and thus be fairly guided, but this guidance will leave in subsequent problems. Mathematica is a bit unlike other programming languages in that its default workspace, the "notebook," is quite like a notebook and not like a regular programming interface. It does read your code top to bottom and compile its Mathematica-syntax into operations for the computer, by pressing Shift+Enter while your text-cursor/carriage is in that block. However, it is very much like a notebook in that you can compile the code you write in small blocks (as marked on the right-hand side of the screen), and edit and change parts separate from one another, much like the entries in a notebook. Mathematica programs are a lot less like files you may compile in Java or C++ and distribute, and a lot more like a page of calculations or workspace.

Lots of very complex commands, not in other programming languages by default, can be guessed in Mathematica, but this is unnecessary. Often if you have something in mind in Mathematica you do not need to guess: all of the functions with examples, options, and errors, can be found on the official Wolfram webpage at http://reference.wolfram.com/language/, as well as built into the program itself under [Help] and then [Find Selected Function]. The best way to learn Mathematica is to try programming in it, and feel around for these commands. Also, generic internet searches for how to do things will often turn up official Mathematica-help pages or useful references on-line. For example, if you are asked to model a curvy set of data points with a function in Mathematica you will need to use a regression method, so just search something like "nonlinear regression in Mathematica" one finds one of the top results is `NonelinearModelFit[]`, which gives use instructions and more.

Learning Goals:

- Practice writing Lagrangians
- Very basic introduction to Mathematica
- Look at applications of harmonic oscillators, damped and undamped, to physics and their normal modes

One of the simplest, and most important, problems in particle physics is the harmonic oscillator; it is jokingly said that the universe is a giant spring mattress in the continuum limit. The harmonic oscillator also demonstrates many of the mathematical techniques that we will need throughout the course and serves as a good introduction to various aspects of programming.

a. Consider a line of four point masses of mass m, connected by springs of spring constant k between adjacent masses, with the endpoints connected to fixed "walls" by springs with spring constant k. Show that the Lagrangian of this system is

$$L = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2\right) - \frac{k}{2}\left(x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_4^2\right) \tag{1.1}$$

and thus using the natural time $\tau = \omega_0 t$, where $\omega_0 = \sqrt{k/m}$, that we may write the equations of motion as below, where a dot now means a derivative with respect to τ, and I is the identity matrix.

$$0 = I\ddot{\mathbf{x}} + \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \mathbf{x} \tag{1.2}$$

More generally, the equation of motion of the N particle analog is

$$0 = I\ddot{\mathbf{x}} + K\mathbf{x} \tag{1.3}$$

where I and K are $N \times N$ matrices. Normal modes correspond to solutions where all oscillators have the same fixed frequency ω, $\mathbf{x}(\tau) = \mathbf{x}_0 e^{i\omega\tau}$. Substituting this ansatz into our formula we get $(K - \omega^2 I)\mathbf{x} = 0$, which has non-trivial solutions for the ω^2 such that $\det(K - \omega^2 I) = 0$.

b. Use Mathematica to program the matrix $K - \omega^2 I$ for an N-particle system for arbitrary N. You *may* want to use the functions `IdentityMatrix[]` for I, `Table[]`, and `KroneckerDelta[]` for defining K. To solve for ω^2 you may need the functions `Det[]`, and `Solve[]` or `NSolve[]`. Use Mathematica to determine

   i. All the ω^2 (with their multiplicities) when $N = 9$ to at least 5 decimal places or in perfect form.

   ii. The highest value of ω^2 when $N = 51$ to at least 5 decimal places. You may want to increase precision on `NSolve[]`.

   iii. Show if A is an arbitrary symmetric matrix, $A = A^T$, and $\mathbf{x}$ and $\mathbf{w}$ are eigenvectors of A corresponding to different eigenvalues, then $\mathbf{x}$ and $\mathbf{w}$ are orthogonal. So that our normal modes truly are normal (orthogonal).

We conclude that any solution may be written as:

$$\mathbf{x}(\tau) = \sum_{i=1}^{N} \left(a_i \cos(\omega_i \tau) + b_i \sin(\omega_i \tau)\right) \mathbf{v}_i \tag{1.4}$$

where $\mathbf{v}_i$ is from the nullspace of $K - \omega_i^2 I$, i.e. an eigenvector corresponding to ω_i. Since our matrices are symmetric we conclude that all the $\mathbf{v}_i$ are orthogonal (unit) vectors. These $\mathbf{v}_i$ are the normal modes.

c. Consider the $N = 4$ particle system.

   i. What are the 4 normal modes and their corresponding frequencies of oscillation? You may want to use the command `NullSpace[]`. Be sure to normalize your vectors, perhaps with the command `Normalize[]`.

   ii. Plot the solutions to the equation of motion for $\tau \in [0, 20]$, using the `Plot[]` command, given the initial conditions

$$\mathbf{x}(0) = \mathbf{w} \qquad \dot{\mathbf{x}}(0) = 0 \tag{1.5}$$

   where $\mathbf{w}$ is the normal mode corresponding to the highest frequency. Your solution should be 1 plot with 4 trajectories on it showing the characteristic vibrational pattern for that mode. You may assume all of the springs are two units of length long at equilibrium so that when you plot the $x_i(\tau)$ on the same graph you can prevent the solutions from overlapping, i.e. send $x_i(\tau) \to x_i(\tau) + 2i$ or something of that nature, to view it in the "global coordinate system" for the problem as opposed to 4 local equilibrium coordinates.

d. Consider the extension of (1.3) to a general damped system

$$0 = M\ddot{\mathbf{x}} + B\dot{\mathbf{x}} + K\mathbf{x} \tag{1.6}$$

where M, B and K are symmetric. This is an equation you may find in an AC RLC-circuit.

i. Use an ansatz for a set of oscillators at the same frequency $\mathbf{z}(\tau;\omega) = \mathbf{z}_{0\omega}e^{\omega\tau}$ on (1.6), $\omega \in \mathbb{C}$ and $\mathbf{z}(\omega;\tau) \in \mathbb{C}^N$. Conclude that we will have non-trivial solutions for the frequency/vector pair $(\omega, \mathbf{z}(\tau;\omega))$ only if $\det(-\omega^2 M + \omega B + K) = 0$. Furthermore, show if $(\omega, \mathbf{z}(\tau;\omega))$ is a solution then so is $(\omega^*, \mathbf{z}(\tau;\omega)^*)$.

We now have $2N$ modes which come in N natural pairs. Unfortunately, they're all complex, and we know that our physical system should only involve real numbers. However, we can turn these $2N$ complex modes into $2N$ real modes using our natural pairings.

ii. Suppose we have the solution $(\omega, \mathbf{z})$ where $\mathbf{z} = \mathbf{z}_0 e^{\omega\tau}$. Show if we expand ω and $\mathbf{z}_0$ into completely real components, $\omega = \alpha + i\beta$ and $\mathbf{z}_0 = \mathbf{x}_0 + i\mathbf{y}_0$, that our $2N$ modes can be written

$$\mathbf{z} = e^{\alpha\tau}\left[(\mathbf{x}_0\cos(\beta\tau) - \mathbf{y}_0\sin(\beta\tau)) + i(\mathbf{x}_0\sin(\beta\tau) + \mathbf{y}_0\cos(\beta\tau))\right] \tag{1.7}$$

iii. Since $\mathbf{z}$ and $\mathbf{z}^*$ are solutions, by linearity, so are arbitrary linear combinations. Conclude that we can turn our $2N$ complex modes into $2N$ real modes via the linear combinations: $(\mathbf{z} + \mathbf{z}^*)/2$ and $(\mathbf{z} - \mathbf{z}^*)/(2i)$, and write them down.

Note our modes with constant frequency are now no-longer normal.

e. Consider the $N = 2$ system, which may occur in the study of circuits, specified by

$$M = \begin{pmatrix} 8 & 1 \\ 1 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \quad M = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} \tag{1.8}$$

i. What are the damping factors and frequencies of oscillation (α, β)?

ii. Include a plot of a characteristic vibrational patterns as before. What happens as $\tau \to \infty$? What happens as $B \to$ The Zero Matrix? What happens if we replace B by cB and $c \to \infty$?

f. Consider the extension of (1.6) to a general forced system with damping

$$\mathbf{F} = M\ddot{\mathbf{x}} + B\dot{\mathbf{x}} + K\mathbf{x} \tag{1.9}$$

i. Assume $\mathbf{F}$ is harmonic, $\mathbf{F}(t) = \mathbf{F}_0 e^{i\omega\tau}$, then proceed as before for the ansatz $\mathbf{z}(\tau,\omega) = \mathbf{z}_0(\omega)e^{i\omega\tau}$, where $\omega \in \mathbb{R}$. Show that

$$\mathbf{z}(\tau,\omega) = \left[(X\cos(\omega\tau) - Y\sin(\omega\tau)) + i(X\sin(\omega\tau) + Y\cos(\omega\tau))\right]\mathbf{F}_0 \tag{1.10}$$

where X and Y are the real and complex parts respectively of $(K + i\omega B - \omega^2 M)^{-1}$.

ii. Assume $F(t) = \mathbf{F}_0 \sin(\omega\tau)$, so that $\mathbf{x}(\tau,\omega) = \mathrm{Im}(\mathbf{z})$. The total energy of the system is

$$E(\tau,\omega) = \frac{1}{2}\left(\mathbf{x}^T M \mathbf{x}\right) + \frac{1}{2}\left(\mathbf{x}^T K \mathbf{x}\right) \tag{1.11}$$

Plug in the result of part *i.* and average $E(\tau,\omega)$ over one period to get

$$\langle E(\omega)\rangle = \frac{1}{4}\omega^2 \mathbf{F}_0^T (XMX + YMY)\mathbf{F}_0 + \frac{1}{4}\mathbf{F}_0^T (XKX + YKY)\mathbf{F}_0 \tag{1.12}$$

g. One place you may see a model like (1.9) is in a classical model of a crystal, where particles are modeled as point masses on springs. Suppose the crystal consists of $N = 5$ atoms and is being forced according to

$$\mathbf{F}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \sin(\omega\tau) \tag{1.13}$$

as in part f. Use M and K as in part a for the $N = 5$ particle system. Plot $\langle E(\omega)\rangle$ for $\omega \in (0, 3)$, and mark the fundamental frequencies on your plot.

i. Assume B is the zero matrix (it is undamped).

ii. Assume B is

$$B = \frac{1}{40}\begin{pmatrix} 9 & 1 & 0 & 0 & 0 \\ 1 & 9 & 7 & 0 & 0 \\ 0 & 7 & 3 & 5 & 0 \\ 0 & 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.14}$$

To do this, you will need to make the matrix $(K + i\omega B - \omega^2 M)$, and take its inverse and split them into the appropriate X and Y. Then use the average energy formula derived. To multiply two matrices in Mathematica becareful not to write `XY` or `X * Y` but `X.Y` with a period inbetween the two matrices/vectors.

2 Year 2: Kapitza's Pendulum

Learning Goals:

- Practice writing Lagrangians and bringing them to dimensionless form suitable for computer use
- Recognize that even extremely simple systems have non-analytic solutions and can quickly run into chaotic behaviour.
- Practice series expansions and approximation techniques
- Gain experience using Mathematica and its basic functions, and reading the included Mathematica documentation. Functions include `Expand[ ]`, `D[ ]`, \. →, `NDSolve[ ]`, `Plot[ ]`, `Manipulate[ ]`, and `ParametricPlot[ ]`.

Most realistic and interesting problems involve non-linear oscillations or behaviour. Consider the following elementary superposition of linear oscillators and how it creates a dramatic non-linear system.

A mass m is on a rigid rod of length L in a standard-orientation gravitational field of strength g and connected to a pivot, forming a pendulum. The pivot is free to move vertically according to some motion $Y(t)$. Let the angle between the downward vertical (measured counter-clockwise) be denoted by θ.

a. Write the Lagrangian for this system. Show that the resulting equation of motion is

$$\ddot{\theta}(t) + \left(\omega_0^2 + \frac{1}{L}\ddot{Y}(t)\right)\sin(\theta(t)) = 0 \tag{2.1}$$

where $\omega_0 = \sqrt{g/L}$. You may do this by hand or use Mathematica to perform the expansions with `Expand[ ]` and derivatives with `D[ ]`.

b. Suppose the driving motion is $Y(t) = A\sin(\omega t)$. To make inputs suitable for a computer, use the following dimensionless substitutions: $\tau = \omega_0 t$, $R = A/L$, and $\Omega = \omega/\omega_0$ and show that (2.1) transforms to

$$\frac{d^2\theta}{d\tau^2} + \left(1 - R\Omega^2\sin(\Omega\tau)\right)\sin(\theta(\tau)) = 0. \tag{2.2}$$

You may do this by hand or use Mathematica to perform the substitutions using the replacement function.

A traditional pendulum ($Y(\tau) = 0$) has two equilibrium points, at $\theta = 0$ and $\theta = \pi$, the one at 0 is stable and at π is unstable. We see our modified pendulum also has equilibrium points at 0 and π. In either case, the equation of motion (2.2) is not solvable analytically. If the mass is above the vertical, $\pi/2 \leq \theta \leq 3\pi/2$, we would expect it to immediately fall. However, if we imagine we start the mass close to π, and oscillate the pivot quickly ($\Omega \gg 1$) and a short distance ($R \ll 1$) that the rod and oscillations will repeatedly "throw" the falling mass back up.

c. Use Mathematica's `NDSolve[ ]` (Numerical Differential Equation Solver) and `Plot[ ]` functions to plot the solutions to (2.2) for the following (in each case, use $\theta'(\tau) = 0$) for at least two periods:

   i. $R = 0.1$, $\Omega = 20$, $\theta(0) = 0.2$.

ii. $R = 0.1$, $\Omega = 10$, $\theta(0) = \pi - 0.2$. Describe in one sentence what is happening to $\theta(t)$.

iii. $R = 0.01$, $\Omega = 20$, $\theta(0) = \pi - 0.2$. Include lines at $y = \pi/2$ and $y = 3\pi/2$. Compare to ii.

it may be beneficial for your understanding of the problem to try your own initial conditions and dimensionless parameters. Include a sketch or a copy of the image with your assignment.

In the case of non-runaway behaviour, θ can be viewed as two separate components, θ_g describing the gradual oscillations about π, and θ_f describing the fast small oscillations about the θ_g position.

d. Using $\theta = \theta_g + \theta_f$ and $|\theta_f| \ll 1$, expand $\sin(\theta)$ in (2.2) about θ_g and separate the equation of motion into the two coupled ODEs

$$\ddot{\theta}_g + \sin\theta_g - R\Omega^2\theta_f \cos\theta_g \sin(\Omega\tau) = 0 \tag{2.3}$$

$$\ddot{\theta}_f + \theta_f \cos\theta_g - R\Omega^2 \sin\theta_g \sin(\Omega\tau) = 0 \tag{2.4}$$

where dots now mean derivatives with respect to dimensionless time, τ. From our plots we know that $\theta_f \propto \sin(\Omega\tau)$, show that

$$\theta_f \simeq [-R\sin\theta_g]\sin(\Omega\tau) \tag{2.5}$$

We say that the gradual motion of θ_g about π is caused by an "effective potential," $V_{eff}(\theta_g)$, so that in the same sense that acceleration is (proportional to) the gradient of a potential, $\ddot{\theta}_g = -dV_{eff}/d\theta_g$.

e. Using (2.5) and (2.3) find $V_{eff}(\theta_g)$. Replace any instance of $\sin^2(\Omega\tau)$ with its average value before calculating $V_{eff}(\theta_g)$. Try out the `Integrate[ ]` function for both the definite and indefinite integrals.

f. What are the conditions on the dimensionless parameters Ω and R for $\theta = \pi$ to be stable? Plot $V_{eff}(\theta_g)$ for cases where π is stable and unstable. What is the maximum possible angle that $\theta \simeq \theta_g$ can be deflected from π?

g. Show in dimensionless coordinates that the canonical momentum of the oscillator is

$$p \propto \dot{\theta} + R\Omega\sin\theta\cos(\Omega\tau) \tag{2.6}$$

Use `ParametricPlot[ ]` and draw at least 4 phase space curves. Make sure at least one has π unstable and one has π stable. Comment on the differences.

h. Show that the Hamiltonian is

$$H(\tau) \propto \dot{\theta}^2 - R^2\Omega^2\cos^2(\Omega\tau) - 2\cos(\theta) + 2R\sin(\Omega\tau) \tag{2.7}$$

and plot it for some (non-trivial) initial parameters. Is it constant? Is this expected from the physical setup of the system?

i. Suppose the pendulum is being driven by a horizontal motion instead, $X(t) = A\sin(\omega t)$, repeat part b with the same constants to get the dimensionless equation of motion

$$\frac{d^2\theta}{d\tau^2} + \sin\theta - \Omega^2 R\cos\theta\sin(\Omega\tau) = 0 \tag{2.8}$$

j. Using $R = 0.1$ and $\Omega = 20$ plot the solution to the new equation of motion. Use `Manipulate[ ]` to vary the initial position $\theta(0)$ and search for unusual stable equilibria in $\theta \in [0, \pi]$, record any that you find. Assume $\dot{\theta}(0) = 0$.

k. Repeat part d, expanding $\sin(\theta)$ and $\cos(\theta)$ using $\theta = \theta_g + \theta_f$ and $|\theta_f| \ll 1$ and separate the ODE into two equations and solve for θ_f as before.

l. Proceed as instructed in parts e and f. Find $V_{eff}(\theta_g)$ and plot it for the distinct cases, compute the exact stable equilibria.

3 Year 2: Spontaneous Symmetry Breaking

Learning Goals:

- Bringing Lagrangians to dimensionless form
- Spontaneous symmetry breaking
- Small oscillations and linearization
- Phase space analysis

Consider a hoop of radius R rotating around the z-axis at constant speed ω with two masses of mass m on opposite sides of the hoop and connected by a spring with spring constant k and equilibrium distance $2r_0$, all under the influence of a gravitational field of strength g.

a. Write a Lagrangian for the system described. Using the substitutions $\omega_0 = \sqrt{2k/m}$, $Z = z/R$, $R_0 = r_0/R$, $\Omega = \omega/\omega_0$, $\tau = \omega_0 t$, and $G = g/R\omega_0^2$, show that the Lagrangian can be brought to the dimensionless form:

$$\tilde{L} = \frac{1}{1-Z^2}\left(\frac{dZ}{d\tau}\right)^2 + \Omega^2(1-Z^2) - 2GZ - \left(R_0 - \sqrt{1-Z^2}\right)^2 \tag{3.1}$$

Recall any Lagrangians are equivalent if they are the same up to a constant or multiple.

b. Define $V_{eff}(Z) = -\Omega^2(1-Z^2) + 2GZ + \left(R_0 - \sqrt{1-Z^2}\right)^2$ and superimpose plots of $V_{eff}(Z)$ for the following initial conditions (try using the `Show[ ]` function):

   i. $G = 0$, $R_0 = 0.4$, $\Omega = 0.600$.

   ii. $G = 0$, $R_0 = 0.4$, $\Omega = 0.774$.

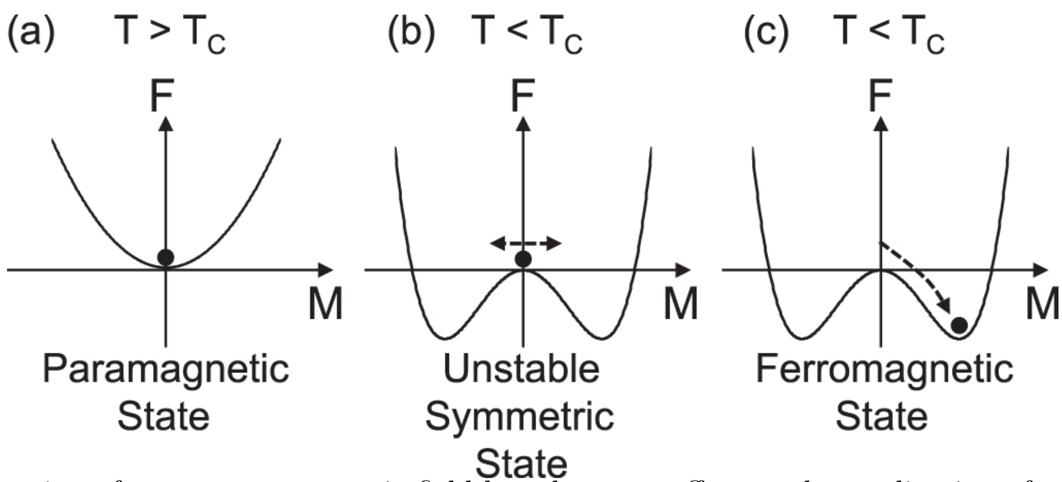
   iii. $G = 0$, $R_0 = 0.4$, $\Omega = 1.000$.

It may be profitable to see the effect when G changes. Find the equilibrium points and classify their stability when the gravitational field strength is $G = 0$, analytically. Solve for the equilibrium points numerically when $G = 0.1$, $\Omega = 0.4$, and $R_0 = 0.4$ and state their stability.

c. Find the equation of motion from $\tilde{L}$. By linearizing the equations of motion find the frequency of small oscillations at the stable equilibrium points when $G = 0$. *Hint: Use the equation of motion and Mathematica's replacement functions to replace Z with $Z_0 + \epsilon\delta Z$ and so on, then use Mathematica to expand the equation of motion in a series in ϵ.*

d. Plot the frequency of small oscillations about the stable equilibria as a function of Ω, for $R = 0.7$ and $G = 0$.

When $G = 0$ and Ω is increased so that the equilibrium points change the system undergoes "spontaneous symmetry breaking." In condensed matter theory ferromagnets undergo spontaneous symmetry breaking below the Curie temperature (in no magnetic field).



The application of a non-zero magnetic field has the same effect as the application of a non-zero G field in our problem, and leads to "explicit symmetry breaking."

In the Standard Model when energies are below the Higgs vacuum expectation value (246GeV), or space is cooler than 10^{15}K, spontaneous symmetry breaking occurs as the electroweak symmetries of the Standard Model Lagrangian break into the weak and electromagnetic interactions. This gives some particles some of their mass, but most importantly gives mass to the W and Z bosons. You may be unsurprised by now if you look for an image of the Higgs potential.

e. Solve the equation of motion you derived in c. for $G = 0$, $R_0 = 0.5$, and the following initial conditions, and explain what's happening briefly. It may be helpful to plot the broken-symmetry equilibrium points.

   i. $Z(0) = 0.7$, $Z'(0) = 0$, $\Omega = 0.6$

   ii. $Z(0) = 0.8275$, $Z'(0) = 0$, $\Omega = 0.6$. Comment on the speed of the particle near $Z = 0$.

   iii. $Z(0) = 0.8275$, $Z'(0) = 0$, $\Omega = 0.8$.

f. Compute both the generalized momentum and the Hamiltonian from the Lagrangian derived in part c. For the initial conditions: $G = 0.03$, $R = 0.5$, $\Omega = 0.6$, $t_{max} = 20\pi$, $Z(0) = 0.7$, and $Z'(0) = 0$ compute the Hamiltonian over time, is it (effectively) constant? Justify.

g. Plot the $p(t)$ versus $Z(t)$ phase space curves (superimposed) for the following initial conditions: $R_0 = 0.5$, $\Omega = 0.6$, $Z'(0) = 0$ and:

   i. $G = 0$, $\Omega = 0.6$, $Z(0) = 0.3, 0.5, 0.82, 0.827, 0.86, 0.9$.

   ii. $G = 0.01$, $\Omega = 0.6$, $Z(0) = 0.3, 0.5, 0.82, 0.827, 0.86, 0.9$.

   ii. $G = 0$, $\Omega = 1/\sqrt{2}$, $Z(0) = 0.3, 0.5, 0.7, 0.8, 0.9$.

Use the `ParametricPlot[ ]` funct ion and set `AspectRatio` $\rightarrow$ `1` to make it easy to read.

4 Year 2: Restricted 3 Body Problem

Learning Goals:

- Bringing Lagrangians to dimensionless form
- Approximation techniques
- Managing many equations/variables in programming (6 coupled ODEs)
- Understanding heliocentric/geocentric/CoM coordinate systems, the behaviour of binary stars, and the restricted 3-body problem, as well as long term stability
- Investigating islands of stability

The gravitational two-body problem, involving two masses who interact only through gravity, is easily solvable given the masses, initial positions, and velocities of two objects. The solutions seem physically intuitive as one experiences similar effects frequently, and typically we learn about the orbits of planets in grade school. The solutions are also geometrically rich from a purely mathematical view since they are all conic-sections.

On the other hand, the gravitational three-body problem is an analytic disaster; each case must be solved independently and numerically even when simplifying assumptions are made. The problem has been studied in various forms by every renaissance mathematician and physicist, and still leads to publications today. The three-body problem is generalizable to the more realistic scenario of n-bodies, but with the difficulty the three-body problem poses (even for modern computers) it is clear that the n-body problems that occur in modeling galaxies are not solved the way we will proceed.

a. Write the Lagrangian for three masses m_1, m_2, m_3 which interact only via gravitational attraction, then show that the 3 equations of motion are

$$\ddot{\vec{r}}_1 + G\left(\frac{m_2}{\|\vec{r}_1 - \vec{r}_2\|^3}(\vec{r}_1 - \vec{r}_2) + \frac{m_3}{\|\vec{r}_1 - \vec{r}_3\|^3}(\vec{r}_1 - \vec{r}_3)\right) = 0 \tag{4.1}$$

$$\ddot{\vec{r}}_2 + G\left(\frac{m_3}{\|\vec{r}_2 - \vec{r}_3\|^3}(\vec{r}_2 - \vec{r}_3) + \frac{m_1}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1)\right) = 0 \tag{4.2}$$

$$\ddot{\vec{r}}_3 + G\left(\frac{m_1}{\|\vec{r}_3 - \vec{r}_1\|^3}(\vec{r}_3 - \vec{r}_1) + \frac{m_2}{\|\vec{r}_3 - \vec{r}_2\|^3}(\vec{r}_3 - \vec{r}_2)\right) = 0 \tag{4.3}$$

Hint 1: Compute the first equation of motion, then using a physical or mathematical argument argue that the other two should take the form above.

Assume that $m_1 > m_2 \gg m_3$ so that m_1 and m_2 orbit their center of mass like a regular two-body problem with average separation distance a, this is called the restricted three-body problem. Further assume that the masses have no initial velocities normal to the plane they lie in originally, so that they will be confined to the same plane for their entire lifetime. Let $M = m_1 + m_2 + m_3 \simeq m_1 + m_2$.

When $m_1 \gg m_2$ Kepler's third law says that the period of m_2 about the CM is $T = 2\pi\sqrt{\frac{a^3}{GM}}$, where a is the average distance between m_1 and m_2. When $m_1 > m_2$ (not $\gg$) the period is slightly different, but this will be a good way to set the scale of dimensionless time.

b. Using $\tau = t/T$, $\vec{w}_i = \vec{r}_i/a$, $\tilde{m}_i = m_i/M$, and $\vec{w}_{ij} = \vec{w}_i - \vec{w}_j$ show that the equations of motion can be rewritten

$$\frac{d^2\vec{w}_i}{d\tau^2} + 4\pi\left(\frac{\tilde{m}_j\vec{w}_{ij}}{w_{ij}^3} + \frac{\tilde{m}_k\vec{w}_{ik}}{w_{ik}^3}\right) = 0 \tag{4.4}$$

for $(ijk) = (123), (231), (321)$, then simplify the two that have m_3 terms using the fact that $m_1 > m_2 \gg m_3$.

c. Consider the following three scenarios, solve the associated restricted three-body problem for each of them, and plot the solutions:

   i. A planet of mass $m_1 = 9.25 \times 10^{26}$kg, a planet of mass $m_2 = 7.5 \times 10^{25}$kg, and a small asteroid are detected at initial positions $\vec{w}_1 = (0, -0.075)$, $\vec{w}_2 = (0, 0.925)$, $\vec{w}_3 = (0, 1.125)$, with initial velocities $\dot{\vec{w}}_1 = (-0.15\pi, 0)$, $\dot{\vec{w}}_2 = (1.85\pi, 0)$, $\dot{\vec{w}}_3 = (2.85\pi, 0)$. Plot the time evolution of the system from the reference frame of an alien on the largest planet for a few τ. Note that the asteroid is "transferred" from planet to planet. When do the first and second transfers occur?

   ii. The aliens living on the largest planet (of the previous problem) have constructed a spherical dome of radius $0.013a$ around the planet which could be damaged by the asteroid if they collide. Current estimates predict asteroid-destroying lasers will be ready in time $\tau = 75$. Should the aliens increase the funding to their asteroid defence program?

   iii. After losing power to all systems (except heat shields) the Millennium Falcon (mass 3) is left stranded near the binary stars Tatoo I (65 solar masses) and Tatoo II (35 solar masses), at initial positions $\vec{w}_1 = (0, -0.2)$, $\vec{w}_2 = (0, 0.8)$, $\vec{w}_3 = (0, 1.1)$, with initial velocities $\dot{\vec{w}}_1 = (1, 0)$, $\dot{\vec{w}}_2 = (6, 0)$, $\dot{\vec{w}}_3 = (0, 0)$. Plot the evolution of the system as seen from the centre of mass coordinate frame for $\tau = 5$. If you'd like, compare it to any other coordinate system to see how "natural" the centre of mass system is.

   iv. Plot the kinetic energy of the Millennium Falcon (you'll have to omit the $m_3/2$ factor) as measured from the center of mass coordinate system, up to $\tau = 12$. We see in the quasi-periodic range a set of sharp peaks (some decaying, and some getting larger), roughly what points in the Falcon's journey would these correspond to? Does the Falcon eventually escape the binary stars? What is this represented by in the graph?

d. Let K be a reference frame that has its origin at the (dimensionless coordinates) center of mass, and has its xy-plane configured so that the 3 masses lie in the xy-plane during their evolution. Further suppose K is rotating about the center of mass with (dimensionless) angular velocity

$$\vec{\Omega} = 2\pi/T\,\hat{z} \tag{4.5}$$

then we see that m_1 and m_2 are stationary in the K frame, and their positions are $r_1 = (-\tilde{m}_2 a, 0, 0)$ and $r_2 = (\tilde{m}_1 a, 0, 0)$ respectively. Show the EoM for m_3 is

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} + 2\Omega\frac{d}{dt}\begin{pmatrix} -y \\ x \end{pmatrix} = -\nabla U_{eff} \tag{4.6}$$

in K, where

$$U_{eff}(x, y) = -\frac{1}{2}\Omega^2(x^2 + y^2) - G(m_1/r_{31} + m_2/r_{32}) \tag{4.7}$$

Hint 2: Use the formula relating lab and body frames in spinning problems derived in the course notes. Then use $F_{net} = F_g = m\vec{a}_3$

Set the scale of distances by letting $a = 1$, and the scale of total mass (but not the important mass ratios) by letting $M = 1$, so distance is given in dimensionless factors of a and M. Let $T = 2\pi$, note how this sets the effective strength of gravity.

e. The Lagrange points are points in the frame K where m_3 can sit at rest, $\nabla U_{eff} = 0$. Show there are two Lagrange points off the x-axis at exactly

$$\left(\frac{1}{2} - m_2, \pm\sqrt{3}\right). \tag{4.8}$$

Then show that the remaining Lagrange points are on the x-axis, and are the roots of a polynomial of degree 5, and so cannot be solved for analytically.[1]

f. Write a program that takes in a mass m_2 and plots the effective potential and computes (and returns) the associated Lagrange points. Have the program superimpose the Lagrange points on the U_{eff} plot. Do it for three systems

   i. The Earth-Moon system.

   ii. The Sun-Jupiter system.

   iii. The binary stars of part c iii.

g. Make a contour plot of the Sun-Jupiter system with the Lagrange Points superimposed, increase the total number of contours and use a plot range of $[-1.4, -1.8]$. You should be able to identify a horse-shoe shaped band in the potential. Often inner moons and space debris are trapped in these orbits in astronomical phenomena. Suppose you will inhabit a small space colony m_3 constructed next to one of the Lagrange points $(x_0 + 0.01, y_0 + 0.01)$, where (x_0, y_0) are the coordinates of the Lagrange point, and plan to live there for 100π time-units, superimpose the trajectory of your colony over the plots you constructed previously. Assuming you want to remain relatively stationary relative to Jupiter and the Sun (stable Lagrange point), where would you want to build? Include a picture of one trajectory starting somewhere unfavourable and one trajectory from a nice location.

[1] By the Abel-Ruffini Theorem.

5 Year 4: Field Theory, Sine-Gordon, and Korteweg-de Vries

Learning Goals:

- Applying Principal of Least Action
- Applying Method of Lines to turn PDEs into coupled ODEs
- Solving coupled ODEs
- See soliton solutions for the first time

The Principal of Extremal Action is always true: the equation of motion is that which minimizes the integral

$$S = \int_\Omega L(\omega, q(\omega), \dots)\, d\omega \tag{5.1}$$

where ω is a collection of parameters and Ω is your available parameter space. In particular, we have considered Lagrangians of the form $L = L(t, q(t), \dot{q}(t))$, so that $\omega = t$, $\Omega = [t_1, t_2]$, and we have shown that the action

$$S = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t))\, dt \tag{5.2}$$

is extremal when $q(t)$ satisfies

$$\frac{dL}{dq} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0\,. \tag{5.3}$$

In field theories instead of having individual particles we have fields $\varphi(\mathbf{x}, t)$ which take on a value everywhere in our spacetime, and we are interested in the value of these fields at different points $(\mathbf{x}, t)$. Analogous to the particle-case, the equation of motion is that which minimizes the integral[2]

$$\mathcal{S} = \int_\Omega \mathcal{L}(\mathbf{x}, t, \varphi(\mathbf{x}, t), \dots)\, d\mathbf{x}\, dt \tag{5.4}$$

a. Suppose we have a Lagrangian density in a $(1+1)$-dimensional spacetime

$$\mathcal{L} = \mathcal{L}(x, t, \varphi(x,t), \partial_x\varphi(x,t), \partial_t\varphi(x,t), \partial_{xx}\varphi(x,t), \partial_{xt}\varphi(x,t), \partial_{tt}\varphi(x,t)) \tag{5.5}$$

Show that the equation of motion for $\varphi(x,t)$ is

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x\varphi)} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t\varphi)} + \partial_{xx} \frac{\partial \mathcal{L}}{\partial(\partial_{xx}\varphi)} + \partial_{xt} \frac{\partial \mathcal{L}}{\partial(\partial_{xt}\varphi)} + \partial_{tt} \frac{\partial \mathcal{L}}{\partial(\partial_{tt}\varphi)} \tag{5.6}$$

where ∂_μ is the derivative with respect to μ, and $\partial_{\mu\mu} = \partial_\mu^2 = \partial_\mu\partial_\mu$.

The Lagrangian for a massive spin-0 scalar field with no interactions is given by the Klein-Gordon Lagrangian

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 - \frac{1}{2}m^2\varphi^2\,. \tag{5.7}$$

The equation of motion is $(\Box^2 + m^2)\varphi = 0$.

[2] $\mathcal{L}$ is called the Lagrangian density for the system, and is related to L for particles by $L = \int \mathcal{L}\, d\mathbf{x}$.

b. Find the equation of motion for the Sine-Gordon Lagrangian

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 \pm m^2\cos\varphi \tag{5.8}$$

which finds applications in high-energy, solid-state, and biological physics.

Suppose we are interested in numerically investigating the evolution of φ over the interval $(x,t) \in [x_{min}, x_{max}] \times [t_{min}, t_{max}]$. In the Method of Lines, we will replace the continuum in x with N_x+1 equally spaced points. Define $\delta x = (x_{max} - x_{min})/N_x$, then our points are at $x_i = x_{min} + i\delta x$ for $i \in \{0, 1, \ldots, N_x\}$. Then we define $\varphi_i(t) = \varphi(x_i, t)$.

c. Show the following substitutions can be made

i. If only even order derivatives occur in the equation of motion, then

$$\partial_{xx}\varphi(x_i,t) \simeq \frac{\varphi_{i+1}(t) - 2\varphi_i(t) + \varphi_{i-1}(t)}{\delta x^2} \tag{5.9}$$

ii. If only odd order derivatives occur in the equation of motion, then

$$\partial_x\varphi(x_i,t) \simeq \frac{\varphi_{i+1}(t) - \varphi_{i-1}(t)}{2\delta x} \tag{5.10}$$

$$\partial_{xxx}\varphi(x_i,t) \simeq \frac{\varphi_{i+3}(t) - 3\varphi_{i+1}(t) + 3\varphi_{i-1}(t) - \varphi_{i-3}(t)}{(2\delta x)^3} \tag{5.11}$$

d. Using part c. and modular boundary conditions ($x_{N_x+1} = x_0$) write the Sine-Gordon equation of motion as N_x+1 coupled ODEs in t. Solve this by any method you choose, and produce $3D$ plots (in x and t coordinates) of

i. $x \in [-20, 20]$, $t \in [0, 30]$, $\varphi(x,0) = \exp(-x^2)$, $\partial_t\varphi(x,0) = 0$, $m = 1$, with plus sign in the Lagrangian.

ii. $x \in [-20, 20]$, $t \in [0, 30]$, $\varphi(x,0) = \exp(-x^2)$, $\partial_t\varphi(x,0) = 0$, $m = 1$, with minus sign in the Lagrangian.

e. Repeat part b. for the (generalized) Korteweg-de Vries Lagrangian

$$\mathcal{L}_{KdV} = \frac{1}{2}\partial_x\psi\partial_t\psi + f(\partial_x\psi) - \frac{1}{2}(\partial_{xx}\psi)^2 \tag{5.12}$$

then make the substitution $\varphi = \partial_x\psi$. This equation was originally used to describe the motion of waves in harbours, and has found applications in solid-state physics, plasma physics, petroleum engineering, geophysics, and atmospheric physics.

f. Repeat part d. for the original Korteweg-de Vries equation ($f(u) = u^3$) for the following conditions

i. $x \in [-20\pi, 20\pi]$, $t \in [0, 20]$, $\varphi(x,0) = \frac{1}{(2\times 0.9999)^2} + cn^2(x/2; 0.9999)$, where cn is the appropriate Jacobi Elliptic function. Comment on the time evolution.

ii. $x \in [-20, 50]$, $t \in [0, 30]$, $\varphi(x,0) = (1/2)\,\mathrm{sech}^2(x/2)$. Comment on anything unusual.

iii. $x \in [-20, 50]$, $t \in [-5, 10]$,

$$\varphi(x,0) = \frac{1 + 2\cosh(x) + \cosh(\sqrt{2}x)}{(2\sqrt{2}\cosh(x/2)\cosh(x/\sqrt{2}) - 2\sinh(x/2)\sinh(x/\sqrt{2}))^2}\,. \tag{5.13}$$

How does this compare to ii?

g. Return to the Sine-Gordon Lagrangian of part b, using the momentum density, $\pi = \frac{\partial \mathcal{L}_{SG}}{\partial(\partial_t \varphi)}$, and the appropriate Legendre transform, $\mathcal{H} = (\partial_t \varphi)\pi - \mathcal{L}$, show that the Hamiltonian density is

$$\mathcal{H}_{SG} = \frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}(\partial_x \varphi)^2 \mp m \cos\varphi \tag{5.14}$$

Then for both sets of initial conditions in part d show that energy (the Hamiltonian) is conserved (pretty well). Note

$$H = \int \mathcal{H}\, dx \tag{5.15}$$

6 Year 4: The Toda Lattice and Hénon-Heiles

Learning Goals:

- Investigate the basic consequences of a Lax pair for integrability
- Derive the Toda lattice and plot its behaviour and a soliton solution
- Derive the the Hénon-Heiles potential from the modular 3 particle Toda lattice
- Show how integrability is lost in Hénon-Heiles

a. (*Lax Pair Implies Integrability.*) The physics of a problem may be described as the solution of some (possibly non-linear) operator. For example, solutions to the wave equation in 1-dimension are those ψ such that $T\psi = 0$ where

$$T = \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) \tag{6.1}$$

Suppose we have two *linear* operators $L(t)$ and $P(t)$ satisfying $L\psi = \lambda\psi$ and $P\psi = \frac{d\psi}{dt}$. Show that λ does not change in time if and only if

$$\frac{dL}{dt} = [P, L] \, . \tag{6.2}$$

Thus we have as many constants of motion (the λ's) as we do degrees of freedom, and so a system in the form (6.2) is integrable.

b. (*Non-Linear Springs.*) Consider a 1-dimensional chain of $N + 1$ identical masses of mass m connected by springs with nearest neighbour interaction potentials V. Denote the positions of the masses by x_i, $i \in \{0, 1, \ldots, N\}$. Define $q_n = x_n - x_{n-1}$ and $q_0 = x_0$.

   i. Show that the Hamiltonian can be written

   $$H = \frac{1}{2m} \sum_{n=0}^{N} (p_n - p_{n+1})^2 + \sum_{n=1}^{N} V(q_n) \tag{6.3}$$

   where we've defined $p_{N+1} = 0$. You can start with $H = T + U$.

   ii. Derive Hamilton's equations for this system.
   Assuming $V'(q)$ is invertible (at least locally), then we have $q_j = -\chi(\dot{p}_j)/m$ for some χ. Taking a time derivative we have

   $$\chi(\dot{p}_j)\ddot{p}_j = -2p_j + p_{j-1} + p_{j+1} \tag{6.4}$$

   just like a regular spring with the χ factor. So the inverse of the derivative of the potential classifies the non-linearity of the interaction.

c. (*Toda's Lattice.*) Consider part b. with $V(a) = e^{-a} + a - 1$. This defines Toda's lattice, which arises in field theory, modelling Langmuir oscillations in plasma physics, modelling conducting polymers, quantum cohomology, and pure mathematics. Assume $m = 1$ throughout.

i. Define

$$L_n = a_n A^+ + a_{n-1}A^- + b_n \tag{6.5}$$
$$P_n = a_n A^+ - a_{n-1}A^- \tag{6.6}$$

where $A^\pm f_n = f_{n\pm 1}$. Show that $\dot{L}_n = [P_n, L_n]$ implies

$$\dot{a}_n = a_n(b_{n+1} - b_n) \tag{6.7}$$
$$\dot{b}_n = 2(a_n^2 - a_{n-1}^2) \tag{6.8}$$

Thus if $a_n = \frac{1}{2}e^{-q_{n+1}/2}$ and $b_n = -\frac{1}{2}(p_n - p_{n+1})$ we recover the regular non-linear lattice equations, so that the Toda Lattice is completely integrable.

ii. Using (q_n, p_n) coordinates and $m = 1$, use Mathematica to program Hamilton's equations and investigate the time evolution of the Toda Lattice. Use N large (at least 100) and plot the time evolution of the lattice assuming

$$q_n(0) = -\log\left[\frac{\left(-e^{2n/5} + e^{2(n+1)/5} + e^{2/5+40\sinh(1/5)}\right)^2}{\left(-e^{2n/5} + e^{2(n+1)/5} + e^{40\sinh(1/5)}\right)\left(-e^{2n/5} + e^{2(n+1)/5} + e^{4/5+40\sinh(1/5)}\right)}\right]$$

and

$$p_n(0) = \frac{\operatorname{sech}(1/5)}{2}\log\left[\frac{-e^{2(n+1)/5} + e^{2(n+2)/5} + \cosh\left(\frac{2}{5} + 40\sinh(\frac{1}{5})\right) + \sinh\left(\frac{2}{5} + 40\sinh(\frac{1}{5})\right)}{-e^{2(n-1)/5} + e^{2n/5} + \cosh\left(\frac{2}{5} + 40\sinh(\frac{1}{5})\right) + \sinh\left(\frac{2}{5} + 40\sinh(\frac{1}{5})\right)}\right]$$

Include a picture of your plot time evolved at least $\Delta t = 50$ units into the future. Describe the behaviour of the solution. What is different about the evolution if $p_n(0) = 0$ for all n?

iii. Do as in the previous problem for $N = 200$ and use the initial conditions $p_n(0) = 0$ and $q_n(0) = \delta_{100,n}$. Include a picture of the plot evolved 50 time units into the future. Describe the behaviour of the solution. Show that energy is conserved in at least the interval $t \in [0, 80]$.

d. (*The Hénon-Heiles Problem.*) Consider a three particle Toda lattice with modular boundary condions. The Hamiltonian is

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{-(\theta_1-\theta_3)} + e^{-(\theta_2-\theta_1)} + e^{-(\theta_3-\theta_1)} - 3 \tag{6.9}$$

i. Derive a new Hamiltonian $H'(\vec{Q}, \vec{P})$ using the canonical transformation from the type-2 generating function

$$F_2(\vec{P}, \vec{\theta}) = P_1\theta_1 + P_2\theta_2 + (P_3 - P_1 - P_2)\theta_3 \tag{6.10}$$

Argue by conservation of total momentum in the original problem that we may set $P_3 = 0$. Derive a new Hamiltonian $H''(x', y', p_x', p_y')$ from $H'(\vec{Q}, \vec{P})$ using the type-2 generating function

$$G_2(Q_1, Q_2, p_x', p_y') = \frac{1}{4\sqrt{3}}\left[(p_x' - \sqrt{3}p_y')Q_1 + (p_x' + \sqrt{3}p_y')Q_2\right] \tag{6.11}$$

Then perform a rescaling to conclude that H has the same dynamics as

$$\bar{H} = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{24}\left(e^{-2\sqrt{3}x+2y} + e^{2\sqrt{3}x+2y} + e^{-4y}\right) - \frac{1}{8} \tag{6.12}$$

ii. Expand $\bar{H}$ in x and y to third order, and keep terms less than or equal to 3 powers in x and y. This is the Hénon-Heiles Hamiltonian. This model originally was used to describe the motion of stars around their galactic center. The model loses its integrability from earlier, except in some very specific cases, showing that the full Toda structure is needed for integrability.

iii. Plot the Poincaré section in the (y, p_y) plane for $E = 0.01, 0.083, 0.12$. Be sure to use many different initial conditions to make sure you fill out the entire Poincaré section. Describe a major difference between the first and second plot versus the third plot. *Hint: It has something to do with the fact that curves in the first two plots are very orderly.*

7 Year 4: The Standard Map

Learning Goals:

- Analyze and apply Poincaré sections and Liouville's theorem to view the qualitative behaviour of a system
- Analyze and apply the KAM theorem to understand the destruction of KAM tori and the relationship with continued fraction expansions
- Apply precise iterative methods to a problem with non-analytic solutions
- Realize connections to integrability of a system and Hamiltonian flows

Consider a free Hamiltonian which is periodically perturbed by a potential $V(q)$ at a rate τ

$$H(p,q,t) = T(p) + V(q) \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \tag{7.1}$$

let q_n and p_n be the position and momentum just *after* the n-th perturbation.

a. Apply Hamilton's equations of motion and show

$$p_{n+1} = p_n - \tau V'(q_{n+1}) \tag{7.2}$$
$$q_{n+1} = q_n + \tau T'(p_n) \tag{7.3}$$

Also show that areas in phase-space are preserved under evolution from $(p_n, q_n) \to (p_{n+1}, q_{n+1})$.

Take $T(p) = p^2/2$, $\tau = 1$, $q = \theta$, and $V(\theta) = k\cos(\theta)$ (assume $k \geq 0$), this could describe a free pendulum that is kicked with a force proportional to its location in its swing.

b. Plot the (θ, p) Poincaré section for this system with modular boundary conditions ($\theta + 2\pi = \theta$ and $p + 2\pi = p$). You can choose the 2π intervals to be centered at 0 or π (each highlights different phenomena). A plot where

   i. There is no perturbation. Comment on how you can tell graphically.
   ii. There are islands of stability and unbroken KAM tori. Describe what is an island of stability on your map, and what is an unbroken KAM tori.
   iii. There are islands of stability but all KAM tori are broken.
   iv. There is chaos. Comment on how you can tell graphically.

   Each plot should have been run on at least 100 different initial conditions under at least 200 iterations of the map. Include the k-values on your plots.

c. Plot the unperturbed Poincaré section for 10 select initial conditions. In particular, ensure that there are initial conditions with both rational and irrational winding numbers.

   i. Just by looking at an unmarked plot, how can you tell which data on the plot correspond to rational and irrational winding numbers? Include your plot to support your claim.

ii. If you slowly increase the strength of the perturbation, which KAM tori break first? Does this align with the predictions of the KAM theorem? Include your plot to support your claim.

iii. Use the KAM theorem and the continued fraction expansion of $\Omega = (\sqrt{5}-1)/2$ to argue that it will be the last KAM torus to break. Include a plot with many different winding numbers to support this.

Consider the standard map except suppose you added a dissipative term $(1-\kappa)$, $\kappa \in [0,1)$, in front of each momentum term, $p \to (1-\kappa)p$.

d. Show for any κ that the fixed points are $(0,0)$ and $(0,\pi)$. Show analytically that $(0,0)$ is always unstable. Show either analytically or graphically (in one picture!) that $(0,\pi)$ is:

   i. An unstable repeller for $0 < k < \frac{2-\kappa}{1-\kappa} - \frac{2}{\sqrt{1-\kappa}}$

   ii. An unstable elliptic point for $\frac{2-\kappa}{1-\kappa} - \frac{2}{\sqrt{1-\kappa}} < k < \frac{2-\kappa}{1-\kappa}$

   iii. A stable elliptic point for $\frac{2-\kappa}{1-\kappa} < k < \frac{2-\kappa}{1-\kappa} + \frac{2}{\sqrt{1-\kappa}}$

   iv. A stable attractor for $\frac{2-\kappa}{1-\kappa} + \frac{2}{\sqrt{1-\kappa}} < k < \infty$

e. For $\kappa = 5/6$ make the plot showcase how as you increase kick strength $(0,\pi)$ goes from an unstable repeller, to an unstable attractor, to an unstable repeller again, and then to a hyperbolic unstable point.

8 Year 4: The Logistic Map

Learning Goals:

- Analyze a prototypical and very simple example of a discrete dynamical system
- Introduce concepts of chaos theory and connections to bifurcation diagrams, fractals, self-similarity, and the Lyapunov exponent
- Investigate islands of stability in a chaotic system

The Logistic Map is a discrete dynamical model with a continuous analogue that models population growth in an ecosystem with finite resources. The discrete model has many different qualitative properties and exhibits many of the fundamental characteristic behaviours of a chaotic system. The map is given by

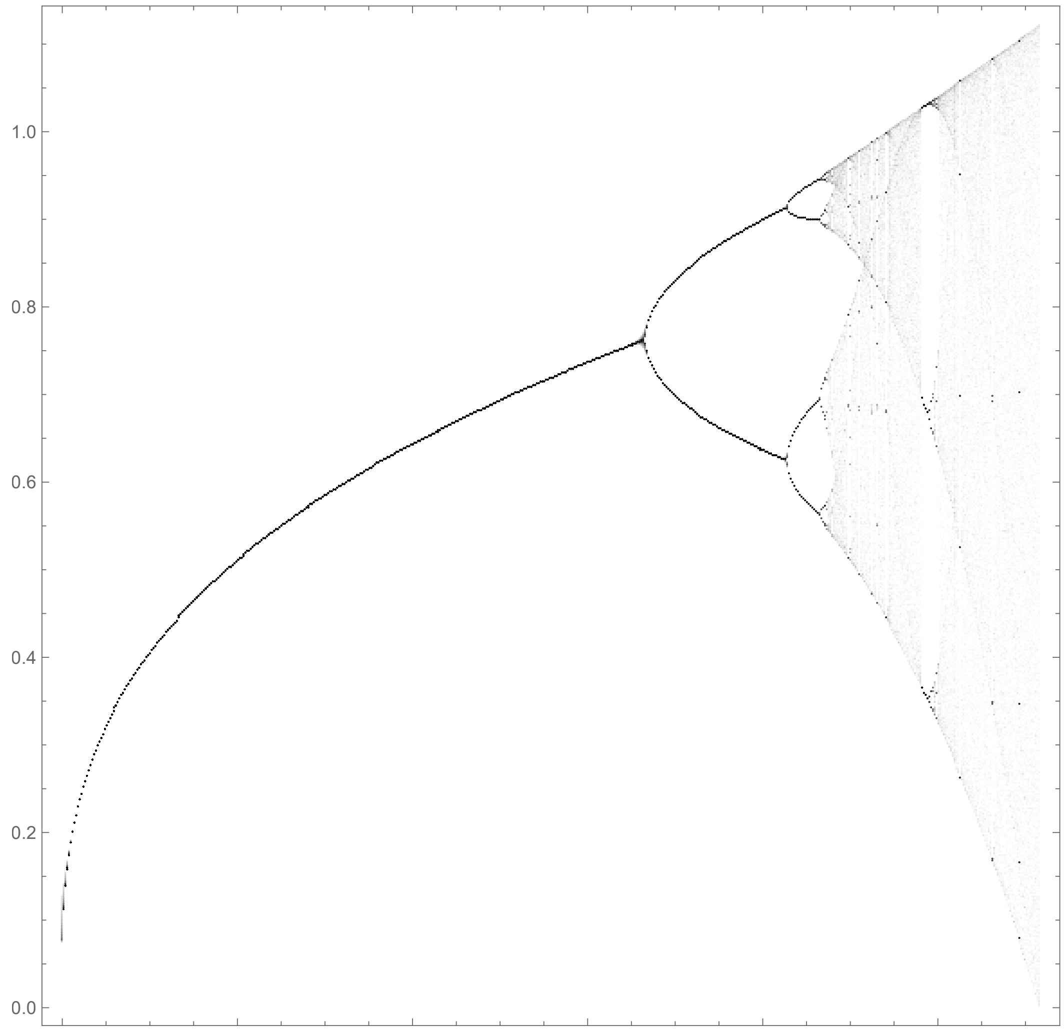
$$x_{j+1} = rx_j(1 - x_j), \quad x_j \in [0, 1], \quad r \in [0, 4] \tag{8.1}$$

One sees that if we allowed $r > 4$ then r causes x to explode as $j \to \infty$ because $x_j(1 - x_j)$ is bounded above by $1/4$. Similarly, if $r \in [0, 1]$ then x would go to 0 as $j \to \infty$ since the powers of r overwhelm the x_j type terms. In fact, this is also the case if $r \in (1, 3]$, that is, $x \to 1 - 1/r$ as $j \to \infty$. The interesting dynamics occurs in the range $r \in (3, 4]$ because the r term effectively brings the quadratic power of x to the same effective strength as the linear part.

a. Using $x_1 = 2/3$ and $r = 2$ make a plot of x_j. Notice how the plot eventually approaches the fixed-point 1/2, after the "transient" behaviour is damped after the first few iterations. Try $r = 2.5$ to see even stronger transient behaviour. Now try $r = 3.2$, notice how (after damping) the plot oscillates between two values, this is a 2-cycle. Try at the value $r = 3.5$, what cycle is this? Do you find any type of cycle at $r = 3.8$? You do not need to include your plots of the cycles.

   The splitting of a 2^n-cycle to a 2^{n+1}-cycle is called a bifurcation. A bifurcation diagram exhibits the evolution of these stable points/cycles, it plots the stable points on the y-axis as a function of the control parameter r on the x-axis.

b. Plot and *include* a bifurcation diagram for $r \in [2.8, 4]$, using $x_1 = 2/3$. For each r you will have to run off a few hundred x_j to be sure you reach steady-state behaviour, then plot a few hundred more x_j ON the plot at that r. This should give just the stable branches (or chaos) on your plot. You will need to choose a very small step size in your parameter as it approaches 3.5. Included for comparison is a bifurcation diagram for the discrete system $x_{j+1} = rx_j - 1.5x_j^4$, which exhibits the same "pitchfork bifurcations" that you should find, as well as eventual descent into chaos.



c. Denote r_n as the point where 2^{n-1} to 2^n bifurcation occurs, ex. $r_1 = 3$. Find as many bifurcation points as you can before the onset of chaos at $r_\infty = 3.5699$, you will need to find at least up to r_4. From the points, compute the ratios of successive bifurcation points to find the "Feigenbaum constant." Include or explain your code/algorithm.

$$\delta = \lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \tag{8.2}$$

One of the miracles of one-dimensional maps is that for any map $x_{i+1} = f(x_i; r)$, regardless of the f, the ratios of successive bifurcations always tends to the same Feigenbaum constant.

Suppose near a 1-cycle we have a point $x_1 = x^* + \delta x$. Expanding to first order around x^* we can write $x_2 = f(x_1) = x^* + f'(x^*)\delta x$, $x_3 = f(f(x_1)) = x^* + f'(f(x^*))f'(x^*)\delta x = x^* + [f'(x^*)]^2\delta x$, and generally $x_n = x^* + [f'(x^*)]^{n-1}\,\delta x$. The point x^* is then stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$.

d. Show that for all x_i^* and x_j^* in an m-cycle that $f^{(m)\prime}(x_i^*) = f^{(m)\prime}(x_j^*)$. Use the result above to declare that an m-cycle is stable if $|f^{(m)\prime}(x^*)| < 1$ and unstable if $|f^{(m)\prime}(x^*)| > 1$. Combine both your results to prove that the Lyapunov exponent, defined as

$$\lambda(r) = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \ln |f'(x_i^*)| \tag{8.3}$$

is negative when there is stability, and positive when there is unstability/chaos. What is the value at a bifurcation?

e. Plot the Lyapunov exponent as a function of r assuming $x_1 = 2/3$. Look at the region $r \in [3.4, 4]$ in particular. Mark a point where a bifurcation happens and a point where chaos begins. Plot it again for the region $r \in [3.54, 3.66]$, what do you notice? (You do not have to include this plot.)

f. You will notice either on the full-scale Lyapunov plot, or the bifurcation diagram, a large island of stability between 3.8 and 3.9. From the bifurcation diagram, we see that this island begins with a 3-cycle. Use this fact to find value where the island of stability begins to at least 3-digits of accuracy, OR compute the exact value.