# Notes for *Gauge Fields, Knots and Gravity*

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Notes taken from the book *Gauge Fields, Knots and Gravity* by John Baez and Javier P. Muniain. 1994 Edition. Notes updated 2018.

# Contents

# 1 Electromagnetism and Differential Geometry

## 1.1 Maxwell's Equations

Classically, the *electric field* $\vec{E}$ and *magnetic field* $\vec{B}$ are (possibly time-dependent) vector functions on "spacetime" which depend on the *electric charge density* $\rho$ and current density $\vec{j}$, satisfying *Maxwell's equations*.

**Result 1.1** (Maxwell's Equations)**.**

$$\begin{aligned} &1.\quad \nabla \cdot \vec{B} = 0 && 3.\quad \nabla \cdot \vec{E} = \rho \\ &2.\quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 && 4.\quad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \end{aligned} \tag{1}$$

**Definition 1.2.** The *Lorentz transformations* of spacetime are the transformations which leave $(x, x) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ invariant.

## 1.2 Manifolds and Topolgy

**Definition 1.3.** A *topological space* is a pair $(X, \tau)$ of a set $X$ and a set $\tau \in \mathcal{P}(X)$. The elements of $\tau$ are called *open sets*, and are required to satisfy:

1. $\emptyset, X \in \tau$
2. If $U, V \in \tau$ then $U \cap V \in \tau$
3. If $U_\alpha \in \tau$ then $\bigcup_{\alpha \in I} U_\alpha \in \tau$ for any index set $I$.

An open set containing $x \in X$ is a *neighbourhood* of $x$. The complement of an open set is called *closed*.

**Definition 1.4.** $f : X \to Y$ is *continuous* if, given any open set $U \subseteq Y$, then $f^{-1}(U) \subseteq X$ is open.

**Note 1.5.** We assume throughout that our topologies are free of pathologies. Namely, they are

1. *Hausdorff*: For any $x, y \in X$ there exists neighbourhoods $U_x$ and $U_y$, of $x$ and $y$ respectively, such that $U_x \cap U_y = \emptyset$.
2. *Paracompact*: Every open cover has a locally finite refinement. That is, each point has a neighbourhood that intersects only finitely many points in the cover.

**Definition 1.6.** Given a topological space $X$, and $U \subseteq X$ open, a *chart* is a continuous function $\varphi : U \to \mathbb{R}^n$ with continuous inverse, $\varphi^{-1} : \varphi(U) \to X$.

**Definition 1.7.** An *n-dimensional manifold* is a topological space, $M$, with charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, where $U_\alpha$ are the open sets covering $M$, and such that the *transition functions*, $\varphi_\alpha \circ \varphi_\beta^{-1}$ defined on the overlaps, are continuous.

If the transition functions are smooth, then $M$ is a *smooth manifold*. We say $f : M \to \mathbb{R}$ is *smooth* if for any $\alpha$, $f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth.

## 1.3 Vector Fields

A vector field on a manifold is, intuitively, a collection of arrows on the surface. The arrows identify directions for directional derivatives, and so we define vector fields algebraically as operators.

**Definition 1.8.** Using $\{x^i\}$ for coordinates on $\mathbb{R}^n$, the *directional derivative* in the direction $v = (v^1, \ldots, v^n)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$vf = v^\mu \partial_\mu f \tag{2}$$

The vector $v = (v^1, \ldots, v^n)$ is thus promoted to the *directional derivative operator*, identifying fields with operators

$$v = v^\mu \partial_\mu \tag{3}$$

**Definition 1.9.** Let $C^\infty(M)$ denote the (commutative) algebra of smooth functions on a manifold $M$. A *vector field* on $M$ is a map $v : C^\infty(M) \to C^\infty(M)$ satisfying:

1. $v(f + g) = v(f) + v(g)$
2. $v(\alpha f) = \alpha v(f)$
3. $v(fg) = v(f)g + fv(g)$

for all $f, g \in C^\infty(M)$. The set of all vector fields is $\mathrm{Vect}(M)$.

**Proposition 1.10.** *If we define addition and multiplication on* $\mathrm{Vect}(M)$ *by*

*1.* $(v + w)(f) = v(f) + w(f)$
*2.* $(gv)(f) = gv(f)$

*for all* $v, w \in \mathrm{Vect}(M)$ *and* $f, g \in C^\infty(M)$*, then* $\mathrm{Vect}(M)$ *is a module over* $C^\infty(M)$*:*

*i.* $f(v + w) = fv + fw$
*ii.* $(f + g)v = fv + gw$
*iii.* $(fg)v = f(gv)$
*iv.* $id_M v = v$

*Proof.* We show just one example: that $(v + w)$ is a vector field. $(v + w)$ clearly satisfies both of the linearity properties of a vector field, so we will just show it satisfies the Leibniz rule:

$$\begin{aligned}(v+w)(fg) &= v(fg) + w(fg)\\ &= v(f)g + fv(g) + w(f)g + fw(g)\\ &= (v+w)(f)g + f(v+w)(g)\end{aligned}$$

□

**Result 1.11.** *The vector fields* $\{\partial_\mu\}$ *on* $\mathbb{R}^n$ *span* $\mathrm{Vect}(\mathbb{R}^n)$ *as a module over* $C^\infty(\mathbb{R}^n)$ *and are linearly independent. Thus every* $v \in \mathrm{Vect}(\mathbb{R}^n)$ *can be written*

$$v = v^\mu \partial_\mu$$

*for some* $v^\mu \in C^\infty(\mathbb{R}^n)$.

**Proposition 1.12.** $v^\mu \partial_\mu = 0$ *iff* $v \equiv 0$.

*Proof.* Suppose $v \equiv 0$, then $v^\mu \partial_\mu f = 0^\mu \partial_\mu f = 0$ for all $f \in C^\infty(\mathbb{R}^n)$. Conversely, if $v^\mu \partial_\mu = 0$, then consider $f = x^i$. Then $0 = v^\mu \partial_\mu x^i = v^i$ for each $i$, so $v \equiv 0$. □

**Corollary 1.13.** *Every vector field* $v$ *on* $\mathbb{R}^n$ *has a unique representation as a linear combination* $v^\mu \partial_\mu$. *Thus* $\{\partial_\mu\}$ *is a basis of* $\mathrm{Vect}(\mathbb{R}^n)$.

### 1.3.1 Tangent Vectors

Given $v$ on $M$ and $f \in C^\infty(M)$, we can evaluate $v(f)$ at any $p \in M$, $v(f)(p)$, to get the result of evaluating the $v$-directional derivative of $f$ at the point $p$.

**Definition 1.14.** A *tangent vector* at $p \in M$ is a function $v_p : C^\infty(M) \to \mathbb{R}$ satisfying:

1. $v_p(f+g) = v_p(f) + v_p(g)$
2. $v_p(\alpha f) = \alpha v_p(f)$
3. $v_p(fg) = v_p(f)g(p) + f(p)v_p(g)$

the set of all tangent vectors at $p$ is the *tangent space* to $p$, $T_pM$.

**Proposition 1.15.** *Any vector field* $v$ *defines a tangent vector* $v_p$ *at any* $p \in M$ *by:*

$$v_p(f) = v(f)(p) \tag{4}$$

*Furthermore, a vector field is completely determined by its tangent vectors on* $M$. *i.e.* $v = w$ *iff* $v_p = w_p$ *for all* $p \in M$.

*Proof.* The fact that any $v$ defines a tangent vector $v_p$ for any $p \in M$ is clear, as the definition of a tangent vector is simply the definition properties of a vector field with evaluation at $p$.

Now we show $v = w$ iff they agree at each point, i.e. $v_p = w_p$ for all $p \in M$.

- $\Rightarrow$ if $v = w$ then $v(f) = w(f)$ for any $f \in C^\infty(M)$; and since these are functions, this just means $v(f)(p) = w(f)(p)$ for any $p \in M$. Since $f$ was arbitrary and $v_p(f) = w_p(f)$ on the whole domain of $v_p$ and $w_p$, so that $v_p = w_p$.
- $\Leftarrow$ Suppose $v_p = w_p$ for any $p \in M$. Then, as functions $C^\infty(M) \to \mathbb{R}$, $v_p$ and $w_p$ agree for all inputs $f$. So $v_p(f) = w_p(f)$ and $v(f)(p) = w(f)(p)$. This means the functions $v(f)$ and $w(f)$ agree for all $p$, so $v(f) = w(f)$. Then since these agree for all $f$, $v = w$.

□

**Result 1.16.** *The converse of the proposition is true: Every tangent vector at $p \in M$ is of the form $v_p$ for some vector field $v \in \mathrm{Vect}(M)$.*

**Note 1.17.** Note that $T_pM$ forms a vector space over $\mathbb{R}$.

**Definition 1.18.** A *curve* is a function $\gamma : \mathbb{R} \to M$ that is *smooth*. i.e. for any $f \in C^\infty(M)$, then $f(\gamma(t))$ depends smoothly on $t$. We define the *tangent vector to the curve* (at some fixed $t$), to be the map $\gamma'(t) : C^\infty(M) \to \mathbb{R}$ such that

$$[\gamma'(t)]\,(f) = \frac{d}{dt} f(\gamma(t)) \tag{5}$$

$\gamma'(t)$ sends functions in the direct that the curve $\gamma$ is moving at time $t$. The fact that $\gamma'(t) \in T_{\gamma(t)}M$ can be seen by noting that $d/dt$ will ensure $\gamma'(t)$ is linear AND satisfies the Leibniz rule at $\gamma(t)$.

### 1.3.2 Covariant versus Contravariant

**Definition 1.19.** Say $\phi : M \to N$ and $f : N \to \mathbb{R}$. We can get a function on $M$ by composition $f \circ \phi$. We define the *pullback* of $f$ from $N$ to $M$ by $\phi$ by

$$\phi^* f = f \circ \phi \tag{6}$$

Thus if $\phi : M \to N$, then the pullback is a map $\phi^* : C^\infty(N) \to C^\infty(M)$.

Because functions $f : N \to \mathbb{R}$ "go backwards" under $\phi : M \to N$, we call functions *contravariant.*

We can generalize the definition of a smoothness from functions and curves to arbitrary maps between manifolds. In particular:

**Definition 1.20.** A function $\phi : M \to N$ is *smooth* if $\phi^* f$ is an element of $C^\infty(M)$ for any $f \in C^\infty(N)$.

We recover the case for smooth functions when $N = \mathbb{R}$, and smooth curves when $M = \mathbb{R}$.

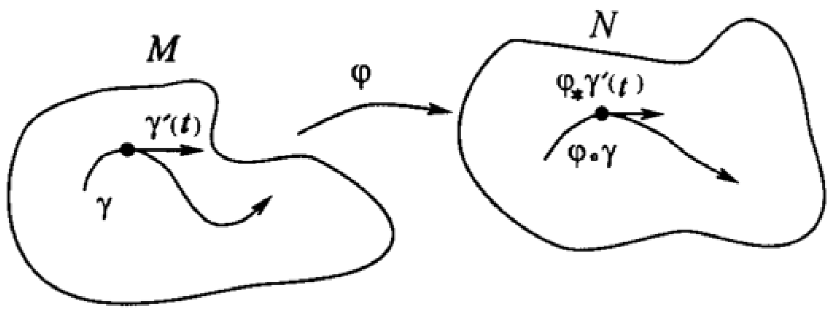
**Definition 1.21.** A tangent vector $v \in T_pM$ and smooth function $\phi : M \to N$ gives a tangent vector on $N$, called the *pushforward* of the vector $v_p$ by $\phi$. The pushforward $\phi_*$ is the map $\phi_* : T_pM \to T_pN$ given by

$$(\phi_* v_p)(f) = v_{\phi(p)}(\phi^* f) \tag{7}$$

**Example 1.22.** Let $M$ and $N$ be manifolds, $\phi : M \to N$, and $\gamma(t)$ a curve on $M$ with tangent $\gamma'(t) \in T_pM$. Then $\phi \circ \gamma$ is a curve in $N$ with tangent vector:

$$(\phi \circ \gamma)'(t) = \phi_*(\gamma'(t)) \in T_{\phi(p)}N \tag{8}$$

That is, the tangent vector to the curve in $N$ (moved from $M$ by $\phi$), is the pushforward of the tangent vector of the original curve in $M$.




To see this, take any function $f \in C^\infty(N)$ and evaluate:

$$\begin{aligned}
(\phi \circ \gamma)'(t)(f) &= \frac{d}{dt}\left[f((\phi \circ \gamma)(t))\right] \\
&= \frac{d}{dt}\left[f(\phi(\gamma(t)))\right] \\
&= \frac{d}{dt}\left[(f \circ \phi)(\gamma(t))\right] \\
&= \frac{d}{dt}\left[\phi^* f(\gamma(t))\right] \\
&= \left[\gamma'(t)\right](\phi^* f) \\
&= \left[\phi_*(\gamma'(t))\right](f)
\end{aligned}$$

Thus far we have only defined the pushforward on individual vectors $v_p \in T_pM$. We can extend the definition in a well-defined way to whole vector fields.

**Definition 1.23.** Let $M$ and $N$ be manifolds, $\phi : M \to N$ a diffeomorphism. Further, let $v$ be a vector field on $M$. There is a well-defined way to define the *pushforward* of a vector field, $\phi_* v : C^\infty(N) \to C^\infty(N)$. Namely by

$$(\phi_* v)_q = \phi_*(v_p) \quad \text{when } q = \phi(p) \tag{9}$$

*Proof.* $q \mapsto (\phi_* v)_q$ is well-defined if $x = y$ implies $(\phi_* v)_x = (\phi_* v)_y$. Well, $x = y$ means $\phi^{-1}(x) = \phi^{-1}(y)$ since $\phi$ is one-to-one (it is a diffeomorphism). Thus we have $(\phi_* v_{\phi^{-1}(x)}) = (\phi_* v_{\phi^{-1}(y)})$ so that $(\phi_* v)_{\phi(\phi^{-1}(x))} = (\phi_* v)_{\phi(\phi^{-1}(y))}$, or rather, $(\phi_* v)_x = (\phi_* v)_y$.

Now we confirm the pushforward is really a smooth vector field on $N$. Since we have defined $(\phi_* v)_{\phi(p)} = (\phi_* v_p)$ we can see pretty easily that $(\phi_* v)$ is linear on the functions $f \in C^\infty(N)$. So we will just confirm the Leibniz rule:

$$\begin{aligned}
(\phi_* v)_\phi(p)(fg) &\equiv (\phi_* v_p)(fg) \\
&= v_p(\phi^*(fg)) \\
&= v_p((fg) \circ \phi) \\
&= v(fg)(\phi(p)) \\
&= v_{\phi(p)}(f)g(\phi(p)) + f(\phi(p))v_{\phi(p)}(g) \\
&= v_p(\phi^* f)g(\phi(p)) + f(\phi(p))v_p(\phi^* g) \\
&= (\phi_* v)_{\phi(p)}(f)g(\phi(p)) + f(\phi(p))(\phi_* v)_{\phi(p)}(g)
\end{aligned}$$

□

### 1.3.3 Flows and the Lie Bracket

**Definition 1.24.** Let $v$ be a vector field on $M$, and suppose $\gamma : \mathbb{R} \to M$ is a curve on $M$ satisfying:

1. $\gamma'(t) = v_{\gamma(t)}$ for all $t \in \mathbb{R}$
2. $\gamma(0) = p$

Then $\gamma$ is the *integral curve* through $p$ defined by $v$. A vector field $v$ is called *integrable* if all integral curves are defined for all $t$.

**Definition 1.25.** Let $\phi_t(p)$ be the integral curve of $v$ through $p \in M$. For any $t \in \mathbb{R}$ define $\phi_t : M \to M$ to be the smooth map (smooth because of theorem on differential equations) at $t$ generated by this integral curve. The family $\{\phi_t\}$ on $M$ is the *flow* generated by $v$.

**Proposition 1.26.** *Let $\phi_t$ be a flow on $M$ generated by some vector field.*

1. *$\phi_0$ is the identity map*
2. *$\phi_t \circ \phi_s = \phi_{t+s}$*

*Proof.* For the first part, $\phi_0(p)$ is the integral curve through $p$ at $t = 0$, defined to be $\phi_0(p) = p$. For the second part, computing, we get

$$\begin{aligned}(\phi_t \circ \phi_s)(p) &= \phi_t(\phi_s(p)) \\ &= \phi_t(\gamma_p(s)) \\ &= \gamma_{\gamma_p(s)}(t) \\ &= \gamma_p(t+s) \\ &= \phi_{t+s}(p)\end{aligned}$$

where $\gamma_p(s)$ denotes the integral curve through $p$ at $s$. □

**Definition 1.27.** Given two vector fields, $v, w \in \mathrm{Vect}(M)$, they define a new vector field $[v, w] \in \mathrm{Vect}(M)$ called the *Lie Bracket* by

$$[v, w](f) = v(w(f)) - w(v(f)) \tag{10}$$

**Proposition 1.28.** *Let $v, w, u \in \mathrm{Vect}(M)$, then the Lie bracket $[v, w]$ satisfies:*

1. $[v, w] \in \mathrm{Vect}(M)$
2. $[w, v] = -[w, v]$
3. $[u, \alpha v + \beta w] = \alpha[u, v] + \beta[u, w]$
4. $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$

*Proof.* The proofs are all straight forward algebra, using the linearity of the vector fields. The only property that doesn't just use linearity is in proving that $\mathrm{Vect}(M)$ is closed under the Lie bracket, and that just needs the Lebiniz rule:

$$\begin{aligned}[v, w](fg) &= v(w(fg)) - w(v(fg)) \\ &= v(w(f)g + fw(g)) - w(v(f)g + fv(g)) \\ &= v(w(f)g) + v(fw(g)) - w(v(f)g) - w(fv(g)) \\ &= v(w(f))g + w(f)v(g) + v(f)w(g) + fv(w(g)) \\ &\quad - w(v(f))g - v(f)w(g) - w(f)v(g) - fw(v(g)) \\ &= (v(w(f)) - w(v(f)))g + f(v(w(g)) - w(v(g))) \\ &= [v, w](f)g + f[v, w](g)\end{aligned}$$

□

So the Lie bracket $[\cdot\,,\cdot]$ is effectively a skew-symmetric trilinear map satisfying the *Jacobi identity*. It measures the failiure of two flows to commute.

## 1.4 Differential Forms

### 1.4.1 1-Forms

**Definition 1.29.** A 1-*form* on a manifold $M$ is a map $\omega : \mathrm{Vect}(M) \to C^\infty(M)$ satisfying

1. $\omega(u + v) = \omega(u) + \omega(v)$
2. $\omega(gv) = g\omega(v)$

for all $u, v \in \mathrm{Vect}(M)$ and $g \in C^\infty(M)$. The space of all 1-forms on $M$ is $\Omega^1(M)$.

**Proposition 1.30.** *If we define addition and multiplication on $\Omega^1(M)$ by*

1. $(\omega + \mu)(v) = \omega(v) + \mu(v)$
2. $(g\omega)(v) = g\omega(v)$

*then $\Omega^1(M)$ is a module over $C^\infty(M)$.*

**Definition 1.31.** For and $f \in C^\infty(M)$ we define the *differential of f* or *exterior derivative* to be the map $df : \mathrm{Vect}(M) \to C^\infty(M)$ by

$$df(v) = v(f) \tag{11}$$

**Proposition 1.32.** *$df$ is a 1-form.*

*Proof.* We prove it by verifying the properties of a 1-form. Let $f \in C^\infty(M)$ arbitrary, then

1. $df(v + w) = (v + w)(f) = v(f) + w(f) = df(v) + dv(w)$
2. $df(gv) = (gv)(f) = gv(f) = gdf(v)$

□

**Definition 1.33.** The *differential operator* is the operator $d : C^\infty(M) \to \Omega^1(M)$ defined by

$$d(f) = df \tag{12}$$

**Proposition 1.34.** *For any $f, g, h \in C^\infty(M)$ and $\alpha \in \mathbb{R}$, $d$ satisfies*

1. $d(f + g) = df + dg$
2. $d(\alpha f) = \alpha df$
3. $(f + g)dh = f\,dh + g\,dh$
4. $d(fg) = df\,g + f\,dg$

*Proof.* 1 through 3 are easy examples of linearity. For property 4 note:

$$d(fg)(v) = v(fg) = v(f)g + fv(g) = d(f)(v)g + fd(g)(v)$$

□

**Example 1.35.** Differentials work exactly how we think they do. For example:

$$d(\sin x) = \cos x \, dx \tag{13}$$

*Proof.* Let $v \in \text{Vect}(\mathbb{R})$ arbitrary, then we know $v = v^x \partial_x$ for some component $v_x$. Then the LHS is

$$d(\sin x)(v) = v(\sin x) = v^x \partial_x \sin x = v^x \cos x$$

and the RHS is

$$(\cos x dx)(v) = (\cos x)v(x) = (\cos x)v^x \partial_x(x) = v^x \cos x$$

□

**Example 1.36.** If $f(x_1, \ldots, x_n)$ is a function on $\mathbb{R}^n$, then

$$df = \partial_\mu f dx^\mu \tag{14}$$

**Theorem 1.37.** *Let $\{x^\mu\}$ the usual coordinate functions on $\mathbb{R}^n$ and $\{\partial_\mu\}$ the associated basis for $\text{Vect}(\mathbb{R}^n)$. The exterior derivatives $\{dx^\mu\}$ form a basis for $\Omega^1(\mathbb{R}^n)$.*

*Proof.* Any vector field $v \in \text{Vect}(\mathbb{R}^n)$ can be written $v = v^\mu \partial_\mu$ for some $v^\mu$. Now note that $dx^\mu(\partial_\nu) = \partial_\nu(x^\mu) = \delta^\mu_\nu$. Now suppose $\omega$ is any 1-form on $\mathbb{R}^n$, and define

$$\omega_\mu = \omega(\partial_\mu)$$

Then to show that our set spans $\Omega^1(\mathbb{R}^n)$ we just have to show that $\omega$ is in fact a linear combination of $dx^\mu$. In particular, we claim that it is exactly the linear combination:

$$\omega = \omega_\mu dx^\mu \tag{15}$$

To see this, just take any arbitrary $v = v^\nu \partial_\nu$ and note that

$$\omega(v) = \omega(v^\nu \partial_\nu) = v^\nu \omega(\partial_\nu) = v^\nu \omega_\nu$$
$$\omega_\mu dx^\mu(v) = \omega_\mu dx^\mu(v^\nu \partial_\nu) = v^\nu \omega_\mu \delta^\mu_\nu = v^\nu \omega_\nu$$

Now we show that it is a basis, i.e., that the set is linearly independent. Well, suppose $\omega \equiv 0$. Well for any $\partial_\nu$ we have $0 = \omega(\partial_\nu) = \omega_\mu dx^\mu(\partial_\nu) = \omega_\mu \delta^\mu_\nu = \omega_\nu$. So $\omega \equiv 0$ iff $\omega_\nu$ for each component $\nu$. □

### 1.4.2 Cotangent Vectors

**Definition 1.38.** Given a manifold $M$ and $p \in M$, a *cotangent vector*, $\omega_p$, at $p$ is a linear map $\omega_p : T_pM \to \mathbb{R}$. The space of all cotangent vectors (at p) is $T_p^*M$.

**Example 1.39.** If $\omega$ is a 1-form on $M$, one way to define a cotangent vector at any $p \in M$ is by defining, for any $v \in \text{Vect}(M)$:

$$\omega_p(v_p) = \omega(v)(p) \tag{16}$$

**Proposition 1.40.** *Defining $\omega_p : T_pM \to \mathbb{R}$ by $\omega_p(v_p) = \omega(v)(p)$ is well-defined.*

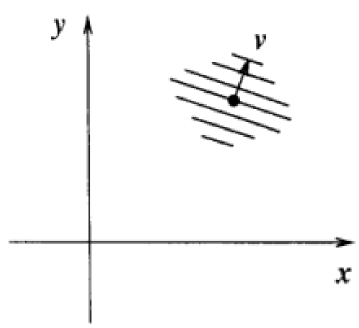
*Proof.* $\omega_p(v_p)$ is well-defined if it depends only on the value of its input vector $v_p \in T_pM$. That is, suppose $v$ and $u$ are two fields on $M$ which agree at $p$, so $v_p = u_p$. Then $\omega_p(v_p) = \omega_p(u_p)$ iff $\omega_p(v_p - u_p) = 0$, which is true because $v_p = u_p$ and $\omega_p$ is linear. □

**Proposition 1.41.** *A 1-form is completely determined by its associated cotangent vectors. That is, $\omega = \mu$ iff $\omega_p = \mu_p$ for all $p \in M$.*

*Proof.* Let $v$ be an arbitrary vector field. Recall that $v$ is determined by its $v_p$. Thus we have as with the vector fields

$$\begin{aligned}
\omega = \mu \iff & \omega(v) = \mu(v) && \text{for all } v \in \text{Vect}(M) \\
\iff & \omega(v)(p) = \mu(v)(p) && \text{for all } v \in \text{Vect}(M) \text{ and } p \in M \\
\iff & \omega_p(v_p) = \mu_p(v_p) && \text{for all } v \in \text{Vect}(M) \text{ and } p \in M \\
\iff & \omega_p = \mu_p && \text{for all } p \in M
\end{aligned}$$

□




In the same way that a tangent vector at a point can be associated with a curve through a point, the cotangent vector can be associated with a function at that point.

In particular, around any point $p$ there are level surfaces defined by a function $f$, and the cotangent vectors are like little infinitessimal level surfaces there. For $v_p \in T_pM$ we have that $df(v_p) = v(f)(p)$ counts how many hyperplanes $v_p$ crosses in the $df$ stack.

**Definition 1.42.** Given any vector space $V$, the *dual*, $V^*$, is the space of all linear functionals $T : V \to \mathbb{R}$. For any map between vector spaces, $f : V \to W$, we have the induced *dual map* $f^* : W^* \to V^*$, given by

$$(f^*w)(v) = w(f(v)) \tag{17}$$

for all $w \in W^*$ and $v \in V$. So $f^* : W^* \to V^*$ is *contravariant.*

**Proposition 1.43.** *Let $V$, $W$, and $X$ be a vector spaces. Let $id : V \to V$ be the identity map on $V$, and $f : V \to W$ and $g : W \to X$ maps on the vector spaces, then*

1. *$(id)^* : V^* \to V^*$ is the identity on $V^*$.*
2. *$(gf)^* = f^*g^*$*

*Proof.* Begin by noting that $id : V \to V$ is defined so that $id(v) = v$ for all $v \in V$, so then for any $w \in V^*$ we have $((id^*)w)(v) = w(id(v)) = w(v)$. So $(id^*)w = w$ and $id^*$ is the identity on $V^*$. For the second part, note that $gf : V \to X$ so $(gf)^* : X^* \to V^*$. Now let $x \in X^*$ and $v \in V$ arbitrary, then we see that

$$\begin{aligned}((gf)^*x)(v) &= x((gf)(v)) \\ &= x(g[f(v)]) \\ &= (g^*x)[f(v)] \\ &= [f^*(g^*x)](v) \\ &= [(f^*g^*)(x)](v)\end{aligned}$$

□

We have a mechanism for pushing forward individual vectors on a manifold, and this in turn gives us a well defined pushforward of vector fields. We can naturally pullback our cotangent vectors globally as follows, and this will give us a way to pullback entire 1-forms.

**Definition 1.44.** Let $\phi : M \to N$, $\phi(p) = q$, and recall that the pushforward is a linear map $\phi_* : T_pM \to T_qN$. The dual to the pushforward of tangent vectors is the *pullback of cotangent vectors*, $\phi^* : T_q^*N \to T_p^*M$. It is given by

$$(\phi^*\omega_q)(v_p) = \omega_q(\phi_*v_p) \tag{18}$$

for $\omega \in T_q^*N$, $v \in T_pM$, $p \in M$ all arbitrary, and $q = \phi(p)$.

That is, we have a natural/forced way to define the pullback of cotangent vectors, from how we defined the pushforward of tangent vectors.

**Theorem 1.45.** *There exists a unique pullback operation for* 1*-forms,* $\phi^* : \Omega^1(N) \to \Omega^1(M)$ *by*

$$(\phi^*\omega)_p = \phi^*(\omega_q) \tag{19}$$

*Proof.* For existence, note that a 1-form $\phi^*\omega$ (whatever it may be) is defined by its points. Thus we can take any $p \in M$ and $v \in \text{Vect}(M)$ arbitrary, and recalling that $v$ is also defined by its points, we have (with the assistance of our definition of the pullback of a cotangent vector and pushforward of tangent vectors) that

$$\begin{aligned} \text{LHS}: &\quad (\phi^*\omega)_p(v_p) \equiv [(\phi^*\omega)(v)](p) = [\omega(\phi_* v)](q) = \omega_q(\phi_* v)_q \\ \text{RHS}: &\quad [\phi * (\omega_q)](v_p) = \omega_q(\phi_* v_p) = \omega_q(\phi_* v)_q \end{aligned}$$

For uniqueness, suppose there were two possible outputs for the pullback of a cotangent vector, $\mu_1$ and $\mu_2$. Then let $\mu = \mu_1 - \mu_2$, and evaluate an arbitrary vector $v \in \text{Vect}(M)$ at any $p \in M$. Then $\mu_p(v_p) = 0$ by linearity, so $\mu = 0$ and thus $\mu_1 = \mu_2$. □

**Theorem 1.46.** *The pullback of the differential is the differential of the pullback.*

$$\phi^*(df) = d(\phi^* f) \tag{20}$$

*Proof.* These are both 1-forms, so it suffices to show that they agree for any $p \in M$, that is, that they produce the same cotangent vector at $p$ for arbitrary $p$. In equations, $\phi^*(df) = d(\phi^* f)$ iff $[\phi^*(df)]_p = [d(\phi^* f)]_p$. Take $v \in \text{Vect}(M)$ arbitrary, since it is determined uniquely by *its* tangent vectors, then

$$\begin{aligned} [\phi^*(df)]_p(v_p) &= [\phi^*(df)_q](v_p) \\ &= (df)_q(\phi_* v_p) \\ &= [(\phi_* v_p)f](q) \\ &= [v_p(\phi^* f)](p) \\ &= (d(\phi^* f))_p(v_p) \end{aligned}$$

□

**Example 1.47.** Let $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = \sin t$. Let $dx$ the usual 1-form on $\mathbb{R}$ (target). Then $(\phi^* dx) = d(\phi^* x) = d(x \circ \sin t) = d(\sin t) = \cos t\, dt$.

### 1.4.3 Change of Coordinates

The charts of a manifold are diffeomorphisms, $\varphi : U \subseteq M \to \mathbb{R}^n$, which turn local data into calculations on $\mathbb{R}^n$.

**Definition 1.48.** Let $\{x^\mu\}$ be coordinates on $\mathbb{R}^n$ with associated coordinate vector fields $\{\partial_\mu\}$, then $\varphi$ pullsback the coordinates of $\mathbb{R}^n$ to the manifold $M$ by

$$x^\mu \equiv \varphi^* x^\mu \tag{21}$$

these are the *local coordinates* on $U \subseteq M$. Any function $f : U \subseteq M \to \mathbb{R}^n$ can be written as $f(x^1, x^2, \ldots, x^n)$.

**Definition 1.49.** Since $\varphi$ is a diffeomorphism, it has a continuous inverse, we define the *coordinate vector fields* on $U \subseteq M$, associated to the local coordinates $\{x^\mu\}$ on $U$, by

$$\partial_\mu \equiv (\varphi^{-1})_* \partial_\mu \tag{22}$$

**Corollary 1.50.** *Any vector field $v$ on $U \subseteq M$ can be written as*

$$v = v^\mu \partial_\mu \tag{23}$$

**Definition 1.51.** The *coordinate* 1*-forms* on $U \subseteq M$, associated to the local coordinates $\{x^\mu\}$ on $U$, are

$$dx^\mu \equiv \varphi^* dx^\mu \tag{24}$$

**Proposition 1.52.** *From our theorem, we see that the differentials of the local coordinates are the coordinate* 1*-forms. That is*

$$d(x^\mu|_U) = (dx^\mu)|_U \tag{25}$$

*Thus we have that any* 1*-form on $U$ can be written as*

$$\omega = \omega_\mu dx^\mu \tag{26}$$

**Definition 1.53.** A *passive coordinate transformation* is a change of local coordinates on the chart. We do not 'move' points of space. An *active coordinate transformation* is a diffeomorphism $\phi : M \to M$.

**Theorem 1.54** (Passive Coordinate Transforms)**.**
*Let $\{x^\mu\}$ and $\{x'^\mu\}$ be two sets of coordinates on $\mathbb{R}^n$ with associated bases $\{\partial_\mu\}$*

*and* $\{\partial'_\mu\}$ *the associated bases for* $\mathrm{Vect}(\mathbb{R}^n)$*. Similarly, let* $\{dx^\mu\}$ *and* $\{dx'^\mu\}$ *be the associated basis for* 1*-forms. Then if we let*

$$T^\nu_\mu = \frac{\partial x'^\nu}{\partial x^\mu} \tag{27}$$

*For the vector fields the basis elements and components transform as:*

$$\partial_\mu = T^\nu_\mu\, \partial'_\nu \qquad \textit{and} \qquad v'^\nu = T^\nu_\mu\, v^\mu \tag{28}$$

*For the* 1*-forms the basis elements and components transform as:*

$$dx'^\nu = T^\nu_\mu\, dx^\mu \qquad \textit{and} \qquad \omega_\mu = T^\nu_\mu\, \omega'_\nu \tag{29}$$

*Proof.* Let $v \in \mathrm{Vect}(\mathbb{R}^n)$ arbitrary, then we may write, uniquely, that $v = v^\mu \partial_\mu = v'^\nu \partial'_\nu$, in the respective bases. Then we can relate the bases by *some* transform, call it $T^\nu_\mu$, satisfying

$$\partial_\mu = T^\nu_\mu \partial'_\nu$$

Acting on $x'^\lambda$ we have

$$\partial_\mu(x'^\lambda) = (T^\nu_\mu \partial'_\nu)(x'^\lambda) = T^\nu_\mu \delta^\lambda_\nu = T^\lambda_\mu$$

or rather

$$T^\nu_\mu = \frac{\partial x'^\nu}{\partial x^\mu}$$

proving the first part about basis elements. Now for some field $v$ we have

$$\begin{aligned} v = v'^\nu\, \partial'_\nu &= v^\mu\, \partial_\mu \\ v'^\nu\, \partial'_\nu &= v^\mu\, T^\nu_\mu \partial'_\nu \end{aligned}$$

and since $\{\partial_\mu\}$ forms a basis, we have that

$$v'^\nu = T^\nu_\mu\, v^\mu$$

Now suppose that $\omega \in \Omega^1(\mathbb{R}^n)$ arbitrary, then we may write, uniquely, that $\omega = \omega_\mu dx^\mu = \omega'_\nu dx'^\nu$, in the respective bases. Then we can relate the bases by *some* transform, call it $S^\nu_\mu$, satisfying

$$dx'^\nu = S^\nu_\mu dx^\mu$$

Acting on $\partial_\lambda$ we have

$$\begin{aligned}
(dx'^{\nu})(\partial_\lambda) &= S^{\lambda}_{\mu}(dx^{\mu})(\partial_\lambda) \\
\partial_\lambda(x'^{\nu}) &= S^{\lambda}_{\mu}\partial_\lambda(x^{\mu}) \\
\frac{\partial x'^{\nu}}{\partial x^{\lambda}} &= S^{\nu}_{\mu}\delta^{\mu}_{\lambda}
\end{aligned}$$

so we see that $S^{\nu}_{\mu} = T^{\nu}_{\mu}$ from before, and we have our first result. The second result is as before. □

**Note 1.55.** Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ ($m = n$, but we will use this to track our data). Write $x^1, \ldots, x^m$ for the $\mathbb{R}^m$ coordinates, and $x'^1, \ldots, x'^n$ for the $\mathbb{R}^n$ coordinates. Since we can't take derivatives of primes (non-primes) with respect to non-primes (primes) we use the shorthand:

$$T^{\lambda}_{\mu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} \equiv \frac{\partial}{\partial x^{\mu}}(\phi^* x'^{\nu}) \tag{30}$$

**Theorem 1.56** (Active Coordinate Transforms)**.**
*If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ as above, and $x^{\mu}$ and $x'^{\nu}$ as above, and using our new definition for $T^{\nu}_{\mu}$, then:*

1. *We can pushforward the coordinate vector fields $\partial_\mu$ on $\mathbb{R}^m$*

$$\phi_*(\partial_\mu) = T^{\nu}_{\mu}\,\partial'_{\nu} \tag{31}$$

2. *We can pullback the coordinate 1-forms $dx'^{\nu}$ on $\mathbb{R}^n$*

$$\phi^*(dx'^{\nu}) = T^{\nu}_{\mu}\,dx^{\mu} \tag{32}$$

*Proof.* 1. Consider any $x'^{\lambda}$ of $\mathbb{R}^n$, then

$$\begin{aligned}
\text{LHS}: &\quad (\phi_*\partial_\mu)(x'^{\lambda}) = \partial_\mu(\phi^* x'^{\lambda}) = T^{\lambda}_{\mu} \\
\text{RHS}: &\quad T^{\nu}_{\mu}(\partial'_{\nu})(x'^{\lambda}) = T^{\lambda}_{\mu}
\end{aligned}$$

2. Consider any $\partial_\lambda$, then

$$\begin{aligned}
\text{LHS}: &\quad \phi^*(dx'^{\nu})(\partial_\lambda) = d(\phi^* x'^{\nu})(\partial_\lambda) = \partial_\lambda(\phi^* x'^{\nu}) = T^{\nu}_{\lambda} \\
\text{RHS}: &\quad T^{\nu}_{\mu}dx^{\mu}(\partial_\lambda) = T^{\nu}_{\mu}\delta^{\nu}_{\lambda} = T^{\lambda}_{\mu}
\end{aligned}$$

□

We can generalize our coordinate transformations to non-orthonormal bases of vector fields and 1-forms. But first we use the resulting theorem

**Result 1.57.** *Let* $\{\partial_\nu\}$ *be the coordinate fields associate to local coordinates* $\{x_\nu\}$ *on* $U \subseteq \mathbb{R}^n$. *Define a set of vector fields*

$$e_\mu = T_\mu^\nu \partial_\nu \tag{33}$$

*where* $T_\mu^\nu$ *is some matrix-valued function on* $U$. *Then* $\{e_\mu\}$ *are a basis of fields on* $U$ *iff for each* $p \in U$, $T_\mu^\nu(p)$ *is invertible.*

**Theorem 1.58.** *If* $\{e_\mu\}$ *is a basis for* $\mathrm{Vect}(\mathbb{R}^n)$ *it induces a unique dual basis of* 1*-forms,* $\{f^\mu\}$*, on* $U$ *such that*

$$f^\mu(e_\nu) = \delta_\nu^\mu \tag{34}$$

*Proof.* Let $\{f^\mu\}$ be the set of 1-forms defined by the above equation. Then $\delta_\nu^\mu = f^\mu(e_\nu) = f^\mu(T_\mu^\lambda \partial_\lambda)$ for some transformation $T$. Now we can write the 1-form $f^\mu$ as $f^\mu = S_\delta^\mu dx^\delta$ for some $S$. Thus $\delta_\nu^\mu = S_\delta^\mu T_\nu^\lambda (dx^\delta \partial_\lambda) = S_\delta^\mu T_\nu^\lambda \delta_\lambda^\delta = S_\lambda^\mu T_\nu^\lambda$ so $ST = \mathbb{I}$. $T$ is invertible by our result, so $S$ is unique by uniqueness of inverse. □

**Theorem 1.59** (Generalized Passive Coordinate Transforms)**.**
*Let* $\{e_\mu\}$ *be a basis for* $\mathrm{Vect}(U)$*, where* $U \subseteq \mathbb{R}^n$*, and* $\{f^\mu\}$ *the associated dual basis of* 1*-forms. If we define* $\{e'_\mu\}$ *by*

$$e'_\mu = T_\mu^\nu \, e^\nu \tag{35}$$

*then we have that*

$$f'^\mu = (T^{-1})_\nu^\mu \, f^\nu \tag{36}$$

*In terms of the components we have that if* $v = v^\mu e_\mu = v'^\nu e'_\nu$ *is an arbitrary vector field and* $\omega = \omega_\mu f^\mu = \omega'_\nu f'^\nu$ *is an arbitrary* 1*-form, then the components transform as*

$$v'^\mu = (T^{-1})_\nu^\mu \, e^\nu \tag{37}$$

*and*

$$\omega'_\mu = T_\mu^\nu \, \omega_\nu \tag{38}$$

The proof is identical, so we skip it. This shows that:

$$\begin{aligned} \text{Vectors} &\mapsto \text{Covariant} \\ \text{Components of Vectors} &\mapsto \text{Contravariant} \\ \text{1-Forms} &\mapsto \text{Contravariant} \\ \text{Components of 1-Forms} &\mapsto \text{Covariant} \end{aligned}$$

**Example 1.60.** If an object has value 1 for length when we measure it in the basis where $\hat{e}_x = 1\text{m}$, then in the basis $\hat{e}'_x = 1\text{cm}$, we *divide* the reference scale by 100, but our length *multiplies* by 100. So the length *vector* (in terms of its components) is *contravariant* since it changes opposite to the shift of basis.

### 1.4.4 p-Forms

**Definition 1.61.** Let $V$ be a vector space. The *exterior algebra* over $V$ is $\bigwedge V$, the algebra generated by $v$ with the relation

$$v \wedge w = -w \wedge v \tag{39}$$

Define $\bigwedge^p V$ to be the subspace of $\bigwedge V$ of linear combinations of p-fold products $v_1 \wedge \cdots \wedge v_p$, where $\bigwedge^1 V \equiv V$ and $\bigwedge^0 V \equiv \mathbb{R}$.

**Proposition 1.62.** *Let $V$ be a vector space and $\dim(V) = n$. Then*

1. $\bigwedge V = \bigoplus_p \bigwedge^p V$
2. $\dim(\bigwedge^p V) = C(n,p)$
3. $\dim(\bigwedge V) = 2^n$

*Proof.* The first thing to note is that for $p > n$ that $\bigwedge^p V$ is vanishing, this comes from the antisymmetry. In which case, part one comes basically as the definition of the algebra generated by the relation. The second part comes from the fact that there are $n$ choose $p$ ways to choose the bases for $\bigwedge^p V$. Then we use the fact that the sum of $C(n,p)$ from $p = 0$ to $n$ is $2^n$. □

**Definition 1.63.** We define $\Omega(M)$ to be the algebra of *differential forms* on $M$, generated by $\Omega^1(M)$ over $C^\infty(M)$ with $\omega \wedge \mu = -\mu \wedge \omega$, with only locally finite linear combinations allowed. We also say $\Omega^0(M) \equiv C^\infty(M)$ and define

$$f \wedge \omega = f\omega \tag{40}$$

for any $f \in C^\infty(M)$ and $\omega \in \Omega(M)$. An element of $\Omega^p(M)$ is a $p$-form.

**Definition 1.64.** Given a vector space $V$, $\bigwedge V$ is *supercommutative* if $\omega \in \bigwedge^p V$ and $\mu \in \bigwedge^q V$, then

$$\omega \wedge \mu = (-1)^{pq} \mu \wedge \omega \tag{41}$$

**Note 1.65.** $\Omega(M)$ is clearly supercommutative.

**Theorem 1.66.** *If $\phi : M \to N$ is a map, then there is a unique pullback map $\phi^* : \Omega(N) \to \Omega(M)$ satisfying:*

*1.* $\phi^*(\alpha\omega) = \alpha\phi^*\omega$ *for* $\alpha \in \mathbb{R}$
*2.* $\phi^*(\omega + \mu) = \phi^*\omega + \phi^*\mu$
*3.* $\phi^*(\omega \wedge \mu) = \phi^*\omega \wedge \phi^*\mu$
*which agrees with the usual* 0 *and* 1*-forms.*

*Proof.* The 3 criteria are definitions, and clearly they agree with the definition of a pullback on functions and on 1-forms because of properties 1 and 2. So really we just need to prove uniqueness, and that should probably involve property 3. To see it is unique, note that property 3 shows that we can break down the pullback of a $p$-form into $p$ 1-forms, each of which are unique and well-defined, and so is the wedge of all those $p$ 1-forms, so it could be nothing else. □

### 1.4.5 The Exterior Derivative

**Definition 1.67.** The *differential*, or *exterior derivative*, is the unique set of maps $d : \Omega^p(M) \to \Omega^{p+1}(M)$ satisfying:
1. $d : \Omega^0(M) \to \Omega^1(M)$ by $df(v) = v(f)$
2. $d(\omega + \mu) = d\omega + d\mu$
3. $d(c\omega) = c\, d\omega$ for $c \in \mathbb{R}$
4. $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p\omega \wedge d\mu$ for $\omega \in \Omega^p(M)$ and $\mu \in \Omega(M)$
5. $d(d\omega) = 0$ for any $\omega \in \Omega(M)$

*Proof.* First we show this map $d$ is well-defined, really that it behaves nicely on the equivalent wedge made by reversing the objects in the wedge:

$$d(-\mu \wedge \omega) = -d(\mu \wedge \omega) = -d\mu \wedge \omega + \mu \wedge d\omega = -\omega \wedge d\mu + d\omega \wedge \mu = d(\omega \wedge \mu)$$

For uniqueness, note any 1-form is a locally finite linear combination of those of the form $df$. Then take any differential form $f dg \wedge dh$ and note that

$$d(f dg \wedge dh) = df \wedge (dg \wedge dh) + f \wedge d(dg \wedge dh) = df \wedge dg \wedge dh$$

□

**Proposition 1.68.** *Let* $I$ *be the multi-index standing for the* $p$*-tuple* $(i_1, \ldots, i_p)$ *of distinct integers between* 1 *and* $n$*. So that*

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{42}$$

*Consider the* $p$*-form*

$$\omega = \omega_I dx^I \tag{43}$$

*then the* $(p+1)$*-form* $d\omega$ *is*

$$d\omega = (\partial_\mu \omega_I) dx^\mu \wedge dx^I \tag{44}$$

*Proof.*

$$d\omega = d(\omega_I dx^I) = d\omega_I \wedge dx^I + (-1)^p \omega_I \wedge d(dx^I) = d\omega_I \wedge dx^I = (\partial^\mu \omega_I) dx^\mu \wedge dx^I$$

□

**Theorem 1.69.** *The differential of the pullback is the pullback of the differential. That is, for any $\omega \in \Omega^p(M)$ we have*

$$\phi^*(d\omega) = d(\phi^*\omega) \tag{45}$$

*Proof.* $\phi^*$ is real-linear, so we can consider just the case of $\omega = f_0 df_1 \wedge \cdots \wedge df_p$. Then we have

$$\begin{aligned}
\phi^*(d\omega) &= \phi^*(df_0 \wedge df_1 \wedge \cdots \wedge df_p) \\
&= \phi^* df_0 \wedge \cdots \wedge \phi^* df_p \\
&= d(\phi^* f_0) \wedge \cdots \wedge d(\phi^* f_p) \\
&= d(\phi^* f_0 \wedge d(\phi^* f_1) \wedge \cdots \wedge d(\phi^* f_p)) \\
&= d(\phi^* f_0 \wedge \phi^* d(f_1) \wedge \cdots \wedge \phi^* d(f_p)) \\
&= d(\phi^*(f_0 \wedge df_1 \wedge \cdots \wedge df_p)) \\
&= d(\phi^* w)
\end{aligned}$$

□

### 1.4.6 Recovering Vector Calculus

In $\mathbb{R}^3$ we have the operator $d$ behaves as

$$0 \to \Omega^0(\mathbb{R}^3) \xrightarrow{d_0} \Omega^1(\mathbb{R}^3) \xrightarrow{d_1} \Omega^2(\mathbb{R}^3) \xrightarrow{d_2} \Omega^3(\mathbb{R}^3) \xrightarrow{d_3} 0 \tag{46}$$

We also have that, writing $V = \mathbb{R}^3$, that

$$\begin{aligned}
\dim\left(\bigwedge^2 V\right) &= C(3,2) = 3 \\
\dim\left(\bigwedge^3 V\right) &= C(3,3) = 1
\end{aligned} \tag{47}$$

so then we have

$$\begin{aligned}
\bigwedge^0 V &= C^\infty(\mathbb{R}^3) \\
\bigwedge^1 V &= \mathbb{R}^3 \\
\bigwedge^2 V &\simeq \mathbb{R}^3 \\
\bigwedge^3 V &\simeq C^\infty(\mathbb{R}^3)
\end{aligned} \tag{48}$$

where the choice of isomorphism chooses if we are in a "left" or "right" handed coordinate system.

Consider a 1-form on $\mathbb{R}^3$. Then $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ so

$$\begin{aligned} d\omega &= (\partial_x \omega_x dx + \partial_x \omega_y dy + \partial_x \omega_z dz) \wedge dx \\ &\quad + (\partial_y \omega_x dx + \partial_y \omega_y dy + \partial_y \omega_z dz) \wedge dy \\ &\quad + (\partial_z \omega_x dx + \partial_z \omega_y dy + \partial_z \omega_z dz) \wedge dz \end{aligned} \tag{49}$$

$$\begin{aligned} &= (\partial_y \omega_z - \partial_z \omega_y) dy \wedge dz \\ &\quad + (\partial_z \omega_x - \partial_x \omega_z) dz \wedge dx \\ &\quad + (\partial_x \omega_y - \partial_y \omega_x) dx \wedge dy \end{aligned} \tag{50}$$

Similarly, if we consider a 2-form on $\mathbb{R}^3$, $\omega = \omega_{xy} dx \wedge dy + \omega_{yz} dy \wedge dz + \omega_{zx} dz \wedge dx$, and we get:

$$d\omega = (\partial_z \omega_{xy} + \partial_x \omega_{yz} + \partial_y \omega_{zx}) dx \wedge dy \wedge dz \tag{51}$$

Now, we note that

$$C^\infty(\mathbb{R}^3) \xrightarrow{\nabla} \mathbb{R}^3 \xrightarrow{\nabla \times} \mathbb{R}^3 \xrightarrow{\nabla \cdot} C^\infty(\mathbb{R}^3) \tag{52}$$

so in combination with our work above we get

$$\nabla \sim d_0 \tag{53}$$

$$\nabla \times \sim d_1 \tag{54}$$

$$\nabla \cdot \sim d_2 \tag{55}$$

## 1.5 Rewriting Maxwell's Equations

Maxwell's first pair of equations, without any time dependence, transform to

$$\nabla \cdot \vec{B} = 0 \to \nabla \cdot \vec{B} = 0 \tag{56}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \to \nabla \times \vec{E} = 0 \tag{57}$$

So if we treat $\vec{B} = (B_x, B_y, B_z)$ as a 2-form

$$B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy \tag{58}$$

and $\vec{E} = (E_x, E_y, E_z)$ as a 1-form

$$E = E_x \, dx + E_y \, dy + E_z \, dz \tag{59}$$

Then these first two equations just become

$$dB = 0 \tag{60}$$
$$dE = 0 \tag{61}$$

In the dynamic case, on a general Minkowski spacetime $\mathbb{R}^4$ $(x_0, x_1, x_2, x_3)$, we define

$$F = B + E \wedge dt \tag{62}$$

In local coordinates

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{63}$$

with $F_{\mu\nu}$ the matrix of components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{64}$$

**Note 1.70.** Over $\mathbb{R}^{d+1}$ (where $d$ is the number of spatial dimensions) we can separate any form into space and time components. i.e. if $\omega = \omega_I dx^I$ then

$$d\omega = \partial_\mu \omega^I dx^\mu \wedge dx^I \tag{65}$$
$$= (dt \wedge \partial_t \omega) + \left(\partial_i \omega_I \, dx^i \wedge dx^I\right) \tag{66}$$

We define $d_S \omega = \partial_i \omega_I dx^i \wedge dx^I$ to be this differential of the spacelike components.

**Theorem 1.71.** *Maxwell's first pair of equations are equivalent to*

$$dF = 0 \tag{67}$$

*Proof.* Begin by noting that

$$dF = dB + dE \wedge dt \tag{68}$$
$$= (d_S B + dt \wedge \partial_t B) + (d_S E + dt \wedge \partial_t E) \wedge dt \tag{69}$$
$$= d_S B + (\partial_t B + d_S E) \wedge dt \tag{70}$$

From the first line, we see if $dB = 0$ and $dE = 0$ then $dF = 0$. Conversely, if $dF = 0$ we have from the last line that $d_S B = 0$ and $\partial_t B + d_S E = 0$; which are Maxwell's equations! □

**Note 1.72.** We see on $M = \mathbb{R}^4$ that this formulation is equivalent to Maxwell's first pair of equations. On a general manifold, $M$, we take $dF = 0$ as the definition for Maxwell's first equation.

**Proposition 1.73.** *For any form $\omega$ on $\mathbb{R} \times S$, there is a unique way to write*

$$d\omega = dt \wedge \partial_t \omega + d_S \omega \tag{71}$$

*such that for any local $x^i$ on $S$ and $t = x^0$, that:*

$$d_S \omega = \partial_i \omega_I \, dx^i \wedge dx^I \tag{72}$$

$$dt \wedge \partial_t \omega = \partial_0 \omega_I \, dx^0 \wedge dx^I \tag{73}$$

**Corollary 1.74.** *On a general manifold, $M$, $dF = 0$ is Maxwell's first pair of equations. The preceding proposition shows if we can write $M = \mathbb{R} \times S$, then we can uniquely decompose*

$$F = F_{i0} \, dx^i \wedge dt + \frac{1}{2} F_{ij} dx^i \wedge dx^j \tag{74}$$

*and that we can define $E_i = F_{i0}$ and $B_{ij} = F_{ij}$ so that*

$$E = E_i \, dx^i \tag{75}$$

$$B = \frac{1}{2} B_{ij} \, dx^i \wedge dx^j \tag{76}$$

$$F = B + E \wedge dt \tag{77}$$

### 1.5.1 The Metric

**Definition 1.75.** In Minkowski spacetime (with $c = 1$) we define the *metric* as

$$v \cdot w = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3 \tag{78}$$

Furthermore, if $V$ is a Minkowski space, and $x \in V$ arbitrary, then

1. if $x \cdot x > 0$ then $x$ is *spacelike*.
2. if $x \cdot x < 0$ then $x$ is *timelike*; the velocity of a particle moving slower than $c$ is timelike.
3. if $x \cdot x = 0$ then $x$ is *null* or *lightlike*.

**Definition 1.76.** Let $V$ be a vector space, a *semi-Riemannian metric* is a symmetric bilinear non-degenerate form, $g : V \times V \to \mathbb{R}$. If $g(v, w) = 0$, then $v$ and $w$ are *orthogonal*. Given a metric $g$ on $V$, we may always construct an *orthonormal basis* $\{e_\mu\}$ such that $g(e_\mu, e_\nu) = \pm\delta_{\mu\nu}$. If the number of $+1$ is $p$ and the number of $-1$ is $q$, then $g$ has *signature* $(p, q)$.

**Definition 1.77.** A *metric* $g$ on an arbitrary manifold $M$ assigns to each $p \in M$ a metric $g_p$ on the tangent space $T_pM$ in a smoothly varying way. By *smoothly varying*, we mean if $v$ and $w$ are smooth fields on $M$, then $g_p(v_p, w_p)$ is smooth on $M$.

**Result 1.78.** *Smoothness implies the signature of $g_p$ is constant on any connected component of $M$.*

**Definition 1.79.** If the signature of a metric is

1. $(n, 0)$ then it is *Riemannian*.
2. $(n-1, 1)$ then it is *Lorentzian*.

**Note 1.80.** The easiest way to make a Lorentzian $(3,1)$ manifold, is to take $M = \mathbb{R} \times S$ for $S$ a 3-dimensional Riemannian manifold with metric ${}^3g$ and define

$$g = -dt^2 +^3 g \tag{79}$$

$$g_\mu\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & {}^3g & \\ 0 & & & \end{pmatrix} \tag{80}$$

This metric is *static*, i.e. independent of time.

**Definition 1.81.** We naturally define measures of length of a path $\gamma : [a, b] \to M$ for spacelike curves

$$\int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}dt \tag{81}$$

called the *arclength*. Similarly, for timelike curves the *proper time* is

$$\int_a^b \sqrt{-g(\gamma'(t), \gamma'(t))}dt \tag{82}$$

**Theorem 1.82.** *If $(V, g)$ is a semi-Riemannian vector space, then $V \simeq V^*$ via the isomorphism $v \to g(v, \cdot)$.*

*Proof.* The map $T(v) = g(v, \cdot)$ is clearly a linear map from $V \to \mathbb{R}$, so $T(v) \in V^*$. Now $T(v_1) = T(v_2)$ iff $T(v_1)(u) = T(v_2)(u)$ for all $u$, which happens iff $g(v_1, u) = g(v_2, u)$ iff $g(v_1 - v_2, u) = 0$. Since $u$ was arbitrary and $g$ is non-degenerate, then $v_1 - v_2 = 0$, and so $v_1 = v_2$, and $T$ is one-to-one. Next, let $w \in V^*$ arbitrary, then $w = w_\nu f^\nu$ so $w(e_\mu) = w_\mu$. Then note that if $v = v^\mu e_\mu$ and we let $w^\nu = g(v, e_\nu) =$

$v^\mu g_{\mu\nu}$, and combine it with the invertibility of $g$ (from the non-degeneracy of $g$), we can solve for $v^\mu = g^{-1}_{\mu\nu} w^\nu$. Thus, if $w \in V^*$ arbitrary and we consider

$$v = v^\mu e_\mu = (g^{-1}_{\mu\nu} w_\nu) e_\mu \in V$$

then $T(v) = w \in V*$. □

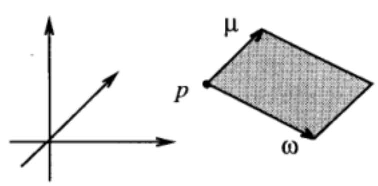
**Corollary 1.83.** *If $(M, g)$ is a semi-Riemannian manifold, then*

1. *For each $p \in M$, $T_pM \simeq T^*_pM$, so each tangent vector $v_p$ is associated with a tiny stack of hyperplanes, perpendicular to the vector.*
2. *There is an isomorphism $\mathrm{Vect}(M) \simeq \Omega^1(M)$, which is how we think of 1-forms as vectors.*

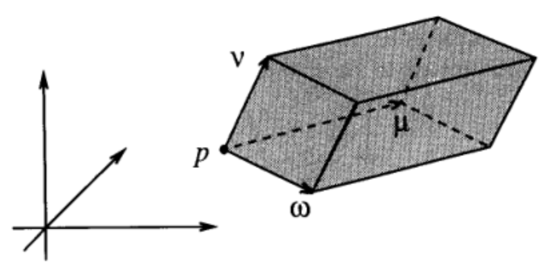
**Note 1.84.** Now that we can associate a 1-form with a vector, we can associate a $p$-form with the parallelepiped formed by the vectors. However, this formulation is not perfect, because

$$(dx + dy) \wedge (dy + dz) = (dy + dz) \wedge (dz - dx) \tag{83}$$

but these do not form the same parallelepiped. The two parallelepipeds lie in the same plane, have the same area, and form a basis with the same orientation (left-handed or right-handed). In fact, it can't be taken super literally because there are elements of $\bigwedge^i T^*_pM$ which are not wedges of $i$ cotangent vectors.




Picture of $\omega \wedge \mu \in \Lambda^2 T^*_p M$




Picture of $\omega \wedge \mu \wedge \nu \in \Lambda^3 T^*_p M$

We convert from upper and lower indices by use of the metric.

**Proposition 1.85.** *Let $\{e_\mu\}$ be a basis of vector fields on a chart. Define the components of the metric as*

$$g_{\mu\nu} = g(e_\mu, e_\nu) \tag{84}$$

*Then non-defeneracy implies the matrix $g_{\mu\nu}$ is invertible. We write the inverse*

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} \tag{85}$$

*This inverse transforms between raised and lowered indices:*

1. *if $v = v^\mu e_\mu$ is a vector field, the corresponding* 1*-form is:*

$$g(v, \cdot) = v_\nu f^\nu \qquad \textit{where} \qquad v_\nu = g_{\mu\nu} v^\mu \tag{86}$$

2. *if $\omega = \omega_\mu f^\mu$ is a* 1*-form, the corresponding vector field is:*

$$\omega^\nu e_\nu \qquad \textit{where} \qquad \omega^\nu = g^{\mu\nu} v_\mu \tag{87}$$

*i.e. $g_\mu \nu$ lowers $v^\mu$ to $v_\nu$, and $g^{\mu\nu}$ raises $\omega_\mu$ to $\omega^\nu$.*

*Proof.* The proof proceeds by direct calculation.

1. The corresponding 1-form to $v = v^\mu e_\mu$ is $g(v, \cdot)$. Now take $e_\nu$, then:

$$\begin{aligned} \text{LHS}: &\quad g(v, e_\nu) = v^\mu\, g(e_\mu, e_\nu) = v^\mu g_{\mu\nu} \\ \text{RHS}: &\quad v_\mu f^\mu(e_\nu) = v_\mu \delta^\mu_\nu = v_\nu \end{aligned}$$

2. Suppose $\omega = \omega_\mu f^\mu$ is a 1-form, and $\bar{\omega} = \bar{\omega}^\mu e_\mu$ is the corresponding vector field, then

$$\begin{aligned} \omega = g(\bar{\omega}, \cdot) &\iff \omega_\mu f^\mu(e_\nu) = g(\bar{\omega}^\mu e_\mu, e_\nu) \\ &\iff \omega_\mu \delta^\mu_\nu = \bar{\omega^\mu} g_{\mu\nu} \\ &\iff \omega_\nu = \bar{\omega}^\mu g_{\mu\nu} \\ &\iff \bar{\omega}^\mu = g^{\mu\nu} \omega_\nu \end{aligned}$$

so the corresponding vector field is $\bar{\omega}^\nu e_\nu$ where $\bar{\omega}^\nu = g^{\mu\nu}\omega_\mu$.

□

**Corollary 1.86.** *For any objects, we have raising and lowering by the metric:*

1. $A_\alpha{}^{\beta\cdots\gamma}{}_{\delta\epsilon\cdots\xi} = g_{\alpha\mu} A^{\mu\beta\cdots\gamma}{}_{\delta\epsilon\cdots\xi}$
2. $A^{\alpha\beta\cdots\gamma\delta}{}_{\epsilon\cdots\xi} = g^{\mu\delta} A^{\alpha\beta\cdots\gamma}{}_{\mu\epsilon\cdots\xi}$
3. $g^\mu_\lambda = g^{\mu\alpha} g_{\alpha_\lambda} = \delta^\mu_\lambda$

The metric allows us to naturally define an *inner-product* on $p$-forms.

**Definition 1.87.** The *inner product* of two $p$-forms $\omega$ and $\mu$ on $M$ is a bilinear function $\langle \omega, \mu \rangle$ on $M$. Since it is bilinear, we can define it on $p$-fold products of 1-forms by:

$$\langle e^1 \wedge \cdots \wedge e^p, f^1 \wedge \cdots \wedge f^p \rangle = \det \left[ \langle e^i, f^j \rangle \right] \tag{88}$$

where the RHS is the determinant of the $p \times p$ matrix whose entries are $\langle e^i, f^j \rangle$ and on 1-forms behaves like

$$\langle \omega, \mu \rangle = g^{\alpha\beta} \omega_\alpha \mu_\beta \tag{89}$$

**Note 1.88.** Note how this inner product on 1-forms is analogous to the inner product on vectors

$$g(v, w) = g_{\alpha_\beta} v^\alpha w^\beta \tag{90}$$

**Theorem 1.89.** *The inner product of p-forms, $\langle \cdot, \cdot \rangle$, is non-degenerate. Furthermore, the wedge products $e^{i_1} \wedge \cdots \wedge e^{i_p}$ form an orthonormal basis of p-forms with*

$$\langle e^{i_1} \wedge \cdots \wedge e^{i_p}, e^{i_1} \wedge \cdots \wedge e^{i_p} \rangle = \epsilon(i_1) \cdots \epsilon(i_p) \tag{91}$$

*where $\epsilon(i) = g(e^i, e^i) = \pm 1$.*

*Proof.* First let $e^I = e^{i_1} \wedge \cdots \wedge e^{i_p}$ and $e^J = e^{j_1} \wedge \cdots \wedge e^{j_p}$ be basis vectors. Obviously each of the $i_k$ is distinct, or else $e^I = 0$, similarly for $e^J$ and it's indices. Now, suppose $e^I$ and $e^J$ agree on $m$ indices, for some $m$. WLOG, since it just changes a sign, we can take these to be the first $m$, so that $i_1 = j_1$, $i_2 = j_2$, ..., and $i_m = j_m$. Then we have that

$$\langle e^{i_1} \wedge \cdots \wedge e^{i_p}, e^{j_1} \wedge \cdots \wedge e^{j_p} \rangle = \det \begin{pmatrix} & & & 0 \\ & M & & \vdots \\ & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

where $M$ is the $m \times m$ matrix whose entries are only along the diagonal and those entries are $\langle e^{i_k}, e^{j_k} \rangle$ at diagonal $k$, giving out result. Non-degeneracy is apparent because $g$ is invertible. □

### 1.5.2 The Volume Form

**Definition 1.90.** Given a vector space, $V$, with bases $\{e_\mu\}$ and $\{f_\mu\}$, there is a unique linear transform $T : V \to V$ such that

$$Te^\mu = f_\mu$$

which is necessarily invertible. We say $\{e_\mu\}$ and $\{f_\mu\}$ have the *same orientation* if $\det(T) > 0$, and *opposite orientation* if $\det(T) < 0$.

**Definition 1.91.** An *orientation* on $V$ is a choice of equivalence classes of bases, where two bases are in the same equivalence class if they have the same orientation.

**Example 1.92.** Even permutations of a basis have the same orientation. Odd permutations of a basis have the opposite orientation.

**Definition 1.93.** Suppose $V$ is an $n$-dimensional vector space with basis $\{e_\mu\}$, then the element

$$e_1 \wedge \cdots \wedge e_n \in \bigwedge^n V \tag{92}$$

is called the *volume element* associated to the basis $\{e_\mu\}$. So that any $\omega \in \bigwedge^n V$ can be written $\omega = c e_1 \wedge \cdots \wedge e_n$ for $c \in \mathbb{R}$

**Proposition 1.94.** *Suppose $\{f_\nu\}$ is a new basis of $V$ such that $f_\nu = T^\mu_\nu e_\mu$. Then the volume element associated to $\{f_\mu\}$ is*

$$f_1 \wedge \cdots \wedge f_n = \det(T)(e_1 \wedge \cdots \wedge e_n) \tag{93}$$

*Proof.* The proof is straightforward algebra and noting the definition of a determinant in terms of sums of permutations weighted by the signs of the permutations. Any terms with duplicate entries will vanish.

$$\begin{aligned} f_1 \wedge \cdots \wedge f_n &= (T^1_1 e_1 + \cdots + T^n_1 e_n) \wedge \cdots \wedge (T^1_n e_1 + \cdots + T^n_n e_n) \\ &= \det(T)(e_1 \wedge \cdots \wedge e_n) \end{aligned}$$

□

**Definition 1.95.** A *volume form* $\omega$ on a manifold $M$ is a nowhere-vanishing $n$-form.

**Example 1.96.** For any $p \in M$, $\omega_p \in T^*_p M$ is a volume element on $T^*_p M$.

**Example 1.97.** The standard volume form on $\mathbb{R}^n$ is $\omega = dx^1 \wedge \cdots \wedge dx^n$.

**Definition 1.98.** $M$ is *orientable* if there exists a volume form on $M$. An *orientation* on $M$ is a choice of equivalence class of volume forms on $M$, where $\omega \sim \omega'$ if $\omega' = f\omega$ for some positive function $f$. We say volume forms int he chosen equivalence class are *positively oriented*, and the others are *negatively oriented*.

**Proposition 1.99.** *Let $M$ be an oriented manifold, then we can cover $M$ with oriented charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, such that the basis $\{dx^\mu\}$ of cotangent vectors on $\mathbb{R}^n$, pulled back to $U_\alpha$ by $\varphi_\alpha$, is positively oriented.*

*Proof.* since $M$ can be oriented, we can define a volume form $\omega$ on $M$ to define the "positive orientation" on $M$. In the chart $U_\alpha$ we can define a volume form by $dx^1 \wedge \cdots \wedge dx^n$. For it to be positively oriented when pulled back, we should have $\omega = c\varphi_\alpha^*(dx^1 \wedge \cdots \wedge dx^n)$ for some $c > 0$, $c \in \mathbb{R}$. Well, we just use whatever our usual charts are, say $\phi_\alpha$, and pull them back. Then, $\omega = c\phi_\alpha^*(dx^1 \wedge \cdots \wedge dx^n)$, and if $c > 0$ let $\varphi_\alpha = \phi_\alpha$, else if $c < 0$ just interchange $dx^1$ and $dx^2$. □

**Definition 1.100.** Given a diffeomorphism $\phi : M \to N$ from one oriented manifold to another, we say $\phi$ is *orientation preserving* if the pullback of any standard orientation basis for the cotangent space of $N$ is a standard orientation basis of the cotangent space of $M$.

**Proposition 1.101.** *If we can cover $M$ with charts such that the transition functions $\varphi_a \circ \varphi_b^{-1}$ are orientation preserving, then $M$ can be made into an oriented manifold.*

*Proof.* In each chart, just pullback the standard volume form on $\mathbb{R}^n$ to $M$. Since each chart is connected by an orientation preserving transition function, the orientation is the same in any chart, and the manifold is oriented everywhere. □

**Theorem 1.102.** *Let $M$ be an oriented $n$-dimensional manifold with metric $g$. Then there is a canonical volume form on $M$ given by*

$$\mathrm{vol} = \sqrt{|\det g_{\mu\nu}|}dx^1 \wedge \cdots \wedge dx^n \tag{94}$$

*for $\{dx^\mu\}$ the local coordinates on $M$.*

*Proof.* Cover $M$ with oriented charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. In any chart set $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ where $\{\partial_\mu\}$ is the pullback of the basis for vector fields from $\mathbb{R}^n$. Well, then vol as defined is a volume form on $U_\alpha$, so now we must show it agrees on all of $M$ by showing that if $\varphi' : U' \to \mathbb{R}^n$ is an overlapping chart with $\varphi : U \to \mathbb{R}^n$ that $\mathrm{vol} = \mathrm{vol}'$ on $U \cap U'$.

Well, on the overlap: $dx'^\nu = T^\nu_\mu dx^\mu$ for the usual $T^\nu_\mu = \partial x'^\nu/\partial x^\mu$. So that we have

$$dx'^1 \wedge \cdots \wedge dx'^n = (\det T)(dx^1 \wedge \cdots \wedge dx^n).$$

This means $\mathrm{vol} = \mathrm{vol}'$ if $\sqrt{|\det g'_{\mu\nu}|} = (\det T)^{-1}\sqrt{|\det g_{\mu\nu}|}$. Well, we have:

$$\begin{aligned} g'_{\mu\nu} &= g(\partial'_\mu, \partial'_\nu) \\ &= g((T^{-1})^\alpha_\mu \partial_\alpha, (T^{-1})^\beta_\nu \partial_\beta) \\ &= (T^{-1})^\alpha_\mu (T^{-1})^\beta_\nu g_{\alpha\beta} \end{aligned}$$

so $\det g'_{\mu\nu} = (\det T)^2 \det g_{\alpha\beta}$. We can take the square root to get our result by noting that $\det T > 0$ since the charts are oriented. □

**Note 1.103.** The volume form is typically denoted

$$\sqrt{|\det g|}\, d^n x \tag{95}$$

or in the Lorentzian case

$$\sqrt{-g}\, d^n x \tag{96}$$

If we don't want to work in the standard basis $\{dx^\mu\}$ for our cotangent spaces, we can generalize it simply.

**Proposition 1.104.** *Let $M$ be an oriented $n$-dimensional semi-Riemannian manifold. Let $\{e_\mu\}$ be any oriented orthonormal basis of cotangent vectors at $p \in M$. Then we have*

$$e_1 \wedge \cdots \wedge e_n = \mathrm{vol}_p \tag{97}$$

*where $\mathrm{vol}_p$ is just the volume form associated to the metric evaluated at $p$ on the manifold.*

*Proof.* If $\{e_\mu\}$ is a standard orientation orthonormal basis, it is related to the standard basis $\{dx^\mu\}$ by an orthogonal transformation, $T$, so $\det T = \pm 1$. But it is orientation preserving so $\det T = 1$. Since in transforming $e_1 \wedge \cdots \wedge e_n$ into the standard basis at $p$ we collect a factor of $\det T$ we get

$$e_1 \wedge \cdots \wedge e_n = (\det T)(dx^1 \wedge \cdots \wedge dx^n)|_p = \mathrm{vol}_p$$

□

]

### 1.5.3 The Hodge Star Operator

The difference in the second pair of Maxwell's equations, besides non-homogeneity, is that $\vec{E}$ now behaves like a 2-form, and $\vec{B}$ is now like a 1-form.

$$\nabla \cdot \vec{E} = \rho \tag{98}$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \tag{99}$$

**Definition 1.105.** Let $M$ be an $n$-dimensional oriented semi-Riemannian manifold, and let the inner product of two $p$-forms $\omega$ and $\mu$ is $\langle \omega, \mu \rangle$. The *Hodge star* is the unique linear map $\star : \Omega^p(M) \to \Omega^{n-p}(M)$ satisfying:

$$\omega \wedge (\star \mu) = \langle \omega, \mu \rangle \,\mathrm{vol} \tag{100}$$

for any $\omega, \mu \in \Omega^p(M)$.

**Note 1.106. To compute the Hodge Star**: Suppose $\{e^\mu\}$ are a basis of positively oriented orthonormal 1-forms on some chart. i.e. $\langle e^\mu, e^\nu \rangle = \delta^\mu_\nu \epsilon(\mu)$. Then for any distinct $1 \leq i_1, \ldots, i_p \leq n$:

$$\star(e^{i_1} \wedge \cdots \wedge e^{i_p}) = \sigma e^{i_{p+1}} \wedge \cdots \wedge e^{i_n} \tag{101}$$

where $\sigma = \operatorname{sgn}(i_1, \ldots, i_n)\epsilon(i_1) \cdots \epsilon(i_p)$, and $\{i_{p+1}, \ldots, i_n\} = \{i_1, \ldots, i_n\} - \{i_1, \ldots, i_p\}$.

**Note 1.107.** Recall that if $W$ is a vector space, the Riesz representation theorem says that every (continuous) linear functional $f \in W^*$ has a unique vector $v \in W$ which emulates its behaviour with the help of the inner product. I.e. for any $w \in W$ that

$$f(w) = \langle w, v \rangle \tag{102}$$

Which gives the isomorphism between $W$ and $W^*$.

The Hodge star is effectively the analogous structure for wedges. If $V$ is an $n$-dimensional vector space with basis $\{e_\mu\}$, then for $0 \leq k \leq n$ the exterior power spaces $\bigwedge^k V$ and $\bigwedge^{n-k} V$ are effectively dual. Take $\lambda \in \bigwedge^k V$ and $\theta \in \bigwedge^{n-k} V$, then taking the exterior product together effectively gives a scalar since it maps to a one-dimensional vector space, i.e. $\lambda \wedge \theta \in \bigwedge^n V$, and so it is a scalar multiple of $e_1 \wedge \cdots \wedge e_n$.

Now fix $\lambda$. There exists a unique linear function $f_\lambda \in \left(\bigwedge^{n-k} V\right)^*$ such that for any $\theta$:

$$\lambda \wedge \theta = f_\lambda(\theta)(e_1 \wedge \cdots \wedge e_n) \tag{103}$$

By construction of $\star\lambda$, it is the element of $\bigwedge^{n-k} V$ such that for any $\theta$ we have

$$f_\lambda(\theta) = \langle \theta, \star\lambda \rangle \tag{104}$$

**Result 1.108.** *Let $V$ an $n$-dimensional vector space with basis $\{e_\mu\}$, and $\lambda \in \bigwedge^k V$ and $\theta \in \bigwedge^{n-k} V$ arbitrary. Then*

$$\lambda \wedge \theta = \langle \theta, \star\lambda \rangle (e_1 \wedge \cdots \wedge e_n) \tag{105}$$

**Theorem 1.109.** *We can formalize our results about vector calculus from earlier*

1. *$\star(\omega \wedge \mu)$ emulates $\omega \times \mu$.*
2. *$\star d\omega$ emulates the curl of $\omega$, $\nabla \times \omega$, for $\omega \in \Omega^1(M)$.*
3. *$\star d \star \omega$ emulates the divergence of $\omega$, $\nabla \cdot \omega$, for $\omega \in \Omega^1(M)$.*

**Theorem 1.110.** *Let $M$ an $n$-dimensional semi-Riemannian oriented manifold with signature $(s, n-s)$. Then $\star^2 : \Omega^p(M) \to \Omega^p(M)$ by*

$$\star^2 = (-1)^{p(n-p)+s} \tag{106}$$

*Proof.* $\omega \wedge \star\omega = \langle \omega, \omega \rangle$ vol and $\star\omega \wedge \star^2\omega = \langle \star\omega, \star\omega \rangle$ vol. Now note that the second equation can be reordered: $\star^2\omega \wedge \star\omega = (-1)^{p(n-p)} \langle \star\omega, \star\omega \rangle$ vol.

Substituting for the vol in both expressions

$$\star^2\omega \wedge \star\omega = \omega \wedge \star\omega \, (-1)^{p(n-p)} \frac{\langle \star\omega, \star\omega \rangle}{\langle \omega, \omega \rangle}$$

so

$$\star^2 = (-1)^{p(n-p)} \langle \star\omega, \star\omega \rangle / \langle \omega, \omega \rangle$$

Now let $\omega$ be a basis $p$-form, $e^{i_1} \wedge \cdots \wedge e^{i_p}$, then $\langle \omega, \omega \rangle = \prod_{j=1}^{p} \epsilon(i_j)$. We also know form our explicit construction of the Hodge star that $\star\omega = \sigma e^{i_{p+1}} \wedge \cdots \wedge e^{i_n}$ so $\langle \star\omega, \star\omega \rangle = \prod_{j=p+1}^{n} \epsilon(i_j)$. Combining these results

$$\begin{aligned} \frac{\langle \star\omega, \star\omega \rangle}{\langle \omega, \omega, \rangle} &= \langle \star\omega, \star\omega \rangle \langle \omega, \omega, \rangle \\ &= \prod_{j=p+1}^{n} \epsilon(i_j) \quad \prod_{j=1}^{p} \epsilon(i_j) \\ &= (-1)^s \end{aligned}$$

which gives our result. □

**Definition 1.111.** Let $M$ an $n$-dimensional oriented semi-Riemannian manifold with signature $(s, n-s)$. Let $\{e^\mu\}$ be an orthonormal basis of 1-forms on a chart. The *Levi-Civita symbol* for $1 \leq i_1, \ldots, i_n \leq n$ is

$$\epsilon_{i_1 \cdots i_n} = \begin{cases} \operatorname{sgn}(i_1, \ldots, i_n) & \text{for all } i_j \text{ distinct} \\ 0 & \text{otherwise} \end{cases} \tag{107}$$

**Theorem 1.112.** *For any p-form*

$$\omega = \frac{1}{p!} \omega_{i_1, \ldots, i_p} e^{i_1} \wedge \cdots \wedge e^{i_p} \tag{108}$$

*we have*

$$(\star\omega)_{j_1, \ldots, j_{n-p}} = \frac{1}{p!} \epsilon^{i_1 \cdots i_p}{}_{j_1 \cdots j_{n-p}} \omega_{i_1 \cdots i_p} \tag{109}$$

*Proof.* For any basis vector

$$\star(e^{i_1} \wedge \cdots \wedge e^{i_p}) = \epsilon_{i_1 \cdots i_n} \epsilon(i_1) \cdots \epsilon(i_p) \, e^{i_{p+1}} \wedge \cdots \wedge e^{i_n} \quad \text{(no sums)}$$

Then

$$
\begin{aligned}
(\star\omega)_{j_1,\cdots,j_{n-p}} &= \langle e^{j_1}\wedge\cdots\wedge e^{j_{n-p}}, \star\omega\rangle\epsilon(j_1)\cdots\epsilon(j_{n-p}) \\
&= \epsilon(j_1)\cdots\epsilon(j_{n-p})\frac{1}{p!}\sum_{i_1,\dots,i_p}\omega_{i_1,\dots,i_p}\langle e^{j_1}\wedge\cdots\wedge e^{j_{n-p}}, \star(e^{i_1}\wedge\cdots\wedge e^{i_p})\rangle \\
&= \epsilon(j_1)\cdots\epsilon(j_{n-p})\frac{1}{p!}\sum_{i_1,\dots,i_p}\omega_{i_1,\dots,i_p}\epsilon_{i_1\cdots i_p}\epsilon(i_1)\cdots\epsilon(i_p)\langle e^{j_1}\wedge\cdots\wedge e^{j_{n-p}}, e^{i_{p+1}}\wedge\cdots\wedge e^{i_n}\rangle \\
&= \epsilon(j_1)\cdots\epsilon(j_{n-p})\frac{1}{p!}\sum_{i_1,\dots,i_p}\omega_{i_1,\dots,i_p}\epsilon_{i_1\cdots i_p}\epsilon(i_1)\cdots\epsilon(i_p)\epsilon(j_1)\cdots\epsilon(j_{n-p})\delta^{j_1\cdots j_{n-p}}_{i_{p+1}\cdots i_n} \\
&= \epsilon(j_1)\cdots\epsilon(j_{n-p})\frac{(-1)^s}{p!}\sum_{i_1,\dots,i_p}\omega_{i_1,\dots,i_p}\epsilon_{i_1\cdots i_p} \\
&= \frac{1}{p!}\epsilon^{i_1\cdots i_p}{}_{j_1\cdots j_{n-p}}\omega_{i_1\cdots i_p}
\end{aligned}
$$

□

### 1.5.4 The Second Pair of Equations

Let $M = \mathbb{R}^4$ the Minkowski spacetime, then we can split $F$ into

$$F = B + E\wedge dt \tag{110}$$

which in component form gives

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{111}$$

With the help of the metric, we have a Hodge star, computing we get

$$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix} \tag{112}$$

**Definition 1.113.** Given a vector *current density*: $\vec{j} = j^1\partial_1 + j^2\partial_2 + j^3\partial_3$ and *charge density* $\rho$, we can form the *current*

$$\vec{J} = \rho\partial_0 + j^1\partial_1 + j^2\partial_2 + j^3\partial_3 \tag{113}$$

or as a 1-form

$$J = j - \rho dt \tag{114}$$

**Theorem 1.114.** *Maxwell's second pair of equations can be written*

$$\star d \star F = J \tag{115}$$

*or if we define $\star_S$ to be the Hodge star on $\mathbb{R}^3$ (or "space") we have*

$$\star_S d_S \star_S E = \rho \tag{116}$$

$$-\partial_t E + \star_S d_S \star_S B = j \tag{117}$$

*Proof.* We have $E = E_j dx^j$ and $B = \frac{1}{2}\epsilon^j{}_{kl} B_j dx^k \wedge dx^l$. Calculating:

$$\begin{aligned}
\star_S E &= \frac{1}{2} E_j \epsilon^j{}_{kl} dx^k \wedge dx^l \\
d_S(\star_S E) &= \frac{1}{2}\epsilon^j{}_{kl}(d_S E_j) \wedge dx^k \wedge dx^l \\
&= \frac{1}{2}\epsilon^j{}_{kl}\partial_m(E_j) dx^m \wedge dx^k \wedge dx^l \\
\star_S d_S \star_S E &= \frac{1}{2}\epsilon^j{}_{kl}\partial_m(E_j) \star (dx^m \wedge dx^k \wedge dx^l) \\
&= \frac{1}{2}\epsilon^j{}_{kl}\partial_m E_j \epsilon^{mkl}\epsilon(x^m)\epsilon(x^k)\epsilon(x^l) \\
&= \frac{1}{2}\epsilon^j{}_{kl}\partial_m E_j \epsilon^{mkl} \\
&= \frac{1}{2}\partial_m E_j (2\delta^m_j) \\
&= \partial_j E_j \\
&= \nabla \cdot \vec{E}
\end{aligned}$$

Similarly, working on $B$ we have

$$\begin{aligned}
\star_S B &= \frac{1}{2}\epsilon^j{}_{kl}B_j \star_S (dx^k \wedge dx^l) \\
&= \frac{1}{2}\epsilon^j{}_{kl}B_j\epsilon^{kl}{}_m dx^m \\
&= B_j dx^j \\
d_S \star_S B &= \partial_m B_j dx^m \wedge dx^j \\
\star_S d_S \star_S B &= \partial_m B_j \star_S (dx^m \wedge dx^j) \\
&= \partial_m B^j \epsilon^{mj}{}_k dx^k \\
&= \epsilon^{ij}{}_k \partial_i B_j dx^k \\
&= \nabla \times \vec{B}
\end{aligned}$$

Now

$$\begin{aligned}
\star F &= \star_S E - \star_S B \wedge dt \\
d \star F &= \star_S \partial_t E \wedge dt + d_S \star_S E - d_S \star_S B \wedge dt \\
\star d \star F &= -\partial_t E - \star_S d_S \star_S E \wedge dt + \star_S d_S \star_S B
\end{aligned}$$

setting $\star d \star F = J$ we have the result. □

**Definition 1.115.** Let $M$ be any spacetime manifold. Then the *electromagnetic field* is a 2-form on $M$, the *current* $J$ is a 1-form on $M$. The *first Maxwell equations* say $dF = 0$. If $M$ is semi-Riemannian and orientable, the *second Maxwell equations* say $\star d \star F = J$.

**Result 1.116.** *To introduce $E$ and $B$ fields we must assume $M = \mathbb{R} \times S$ and write $F = B + E \wedge dt$, and $J = j - \rho dt$. The first pair split into*

$$dF = 0 \implies d_S B = 0 \qquad \textit{and} \qquad \partial_t B + d_S E = 0 \tag{118}$$

*and if* $\dim(S) = 3$ *and there is a static metric $g = -dt^2 +^3 g$ then the second pair splits*

$$\star d \star F = J \implies \star_S d_S \star_S E = \rho \qquad \textit{and} \qquad -\partial_t E + \star_S d_S \star_S B = j \tag{119}$$

**Definition 1.117.** The *vacuum Maxwell equations* are $dF = 0$ and $d\star F = 0$. These equations are preserved by $F \to \star F$.

**Note 1.118.** In 4-dimensional Riemannian space $\star^2 = 1$, so the eigenvalues of $\star$ are $\pm 1$. Thus we can split $F$ into *self-dual* $F_+$, and *anti-self-dual* $F_-$, parts, and write $F = F_+ + F_-$, where $\star F_\pm = \pm F_\pm$, and $F_\pm = \frac{1}{2}(F \pm \star F)$.

In 4-dimensional Lorentzian space, $\star^2 = -1$, so the eigenvalues are $\pm i$. Thus we can define $F_\pm = \frac{1}{2}(F \mp i \star F)$, so that $F = F_+ + F_-$ and $\star F = \pm i F_\pm$.

**Note 1.119.** On Minkowski $M = \mathbb{R}^4$ the vacuum equations give light moving through empty space. $\star F = \star_S E - \star_S B \wedge dt$ so $F$ is self-dual if $\star_S E = iB$ and $\star_S B = -iE$. These conditions hold for all $t$ if

$$E = E_i dx^i \qquad \text{and} \qquad B = \frac{-i}{2}\epsilon^j{}_{kl} E_k dx^k \wedge dx^l \tag{120}$$

**Definition 1.120.** A *planewave* is an equation of the form

$$E(x) = \vec{E} e^{ik_\mu x^\mu} \tag{121}$$

where $\vec{E} = E_j dx^j$ is a constant $\mathbb{C}$-valued 1-form on $\mathbb{R}^3$, and $k \in \mathbb{R}^4 - \{0\}$, is a fixed covector called the *energy-momentum*

**Proposition 1.121.** *If $F$ is self-dual (as in a vacuum in 4D), and the electric field is a plane wave: $E = \vec{E} e^{ik_\mu x^\mu}$. Then*

1. $B = \vec{B} e^{ik_\mu x^\mu}$ *where* $\vec{B} = -i \star_S \vec{E}$
2. $\langle \vec{E}, {}^3k \rangle = 0$, *so $\vec{E}$ is perpendicular to the momentum, and light is a transverse wave.*
3. ${}^3k \wedge \vec{E} = k_0 \vec{B}$, *so the cross product of ${}^3k$ and $\vec{E}$ is proportional to $\vec{B}$ by $k_0$, the frequency of the wave.*
4. *The energy-momentum of light is light-like, i.e.* $k^\mu k_\mu = 0$.

*Proof.* If $F$ is self-dual and its electric field is a plane-wave, then the previous results give that the magnetic field is

$$B(x) = \vec{B} e^{ik_\mu x^\mu}$$

where $\vec{B} = i \star_S \vec{E}$.

The first Maxwell equation holds when

$$\begin{aligned} 0 &= d_S \vec{B} \\ &= \vec{B} \wedge d_S e^{ik_\mu x^\mu} \\ &= \vec{B} \wedge (e^{ik_\mu x^\mu} {}^3k) \\ 0 &= \vec{B} \wedge {}^3k \end{aligned}$$

But in terms of $\vec{E}$, this means $\star_S \vec{E} \wedge {}^3k = 0$ or

$$\langle \vec{E}, {}^3k \rangle = 0$$

In addition, we have

$$\begin{aligned} 0 &= \partial_t B + d_S E \\ &= ik_0 B + d_S(e^{ik_\mu x^\mu} E_j dx^j) \\ &= ik_0 B + ik_l E_j dx^j \wedge dx^l \\ &= ik_0 B - i^3 k \wedge E \end{aligned}$$

which means ${}^3k \wedge \vec{E} = k_0 B$.

To see that the energy-momentum of light is light-like ${}^3k \wedge \vec{E} = k_0 \vec{B} = -ik_0 \star_S \vec{E}$, implying $\langle {}^3k \wedge \vec{E}, {}^3k \wedge \vec{E} \rangle = k_0^2 \langle \star_S \vec{E}, \star_S \vec{E} \rangle$. Expanding the LHS

$$\begin{aligned} \langle {}^3k \wedge \vec{E}, {}^3k \wedge \vec{E} \rangle &= k_l k_{l'}^* E_j E_{j'}^* \langle dx^l \wedge dx^j, dx^{l'} \wedge dx^{j'} \rangle \\ &= k_l k_{l'}^* E_j E_{j'}^* (\delta^{jj'} \delta^{ll'} - \delta^{j'l} \delta^{jl'}) \\ &= \langle {}^3k, {}^3k \rangle \langle \vec{E}, \vec{E} \rangle - \left| \langle {}^3k, \vec{E} \rangle \right|^2 \\ &= \langle {}^3k, {}^3k \rangle \langle \vec{E}, \vec{E} \rangle \end{aligned}$$

Expanding the RHS

$$\begin{aligned} \langle \star_S \vec{E}, \star_S \vec{E} \rangle &= \frac{1}{4} \epsilon^j{}_{kl} \epsilon^{j'}{}_{k'l'} E_j E_{j'}^* \langle dx^k \wedge dx^l, dx^{k'} \wedge dx^{l'} \rangle \\ &= \frac{1}{4} \epsilon^j{}_{kl} \epsilon^{j'}{}_{k'l'} E_j E_{j'}^* (\delta^{kk'} \delta^{ll'} - \delta^{k'l} \delta^{kl'}) \\ &= \langle \vec{E}, \vec{E} \rangle \end{aligned}$$

So $\langle {}^3k, {}^3k \rangle \langle \vec{E}, \vec{E} \rangle = k_0^2 \langle \vec{E}, \vec{E} \rangle$, so that $k^\mu k_\mu = 0$. $\square$

## 1.6 deRham Theory in Electromagnetism

### 1.6.1 Closed and Exact 1-Forms

On an $n$-dimensional manifold, $M$, the exterior derivative defines a (co)-chain complex, or *de Rham complex*:

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-2}} \Omega^{m-1}(M) \xrightarrow{d_{m-1}} \Omega^m(M) \xrightarrow{d_m} 0 \qquad (122)$$

**Definition 1.122.** If $d_{n-1} : \Omega^{n-1}(M) \to \Omega^n(M)$ and $d_n : \Omega^n(M) \to \Omega^{n+1}(M)$, then $\omega \in \Omega^n(M)$ is called

1. *exact* if $\omega \in \text{im}(d_{n-1})$. We write $B^n(M) = \text{im}(d_{n-1})$.
2. *closed* if $\omega \in \ker(d_n)$. We write $Z^n(M) = \ker(d_n)$.

**Theorem 1.123.** $d^2 = 0$

**Corollary 1.124.** *All exact forms are closed. i.e.* $B^n(Z) \subseteq Z^n(Z)$ *for all* $n$*.*

**Definition 1.125.** Let $E$ be a 1-form and $B$ a 2-form. A *scalar potential* is a 0-form (function) $\phi$ such that $E = -d\phi$. A *vector potential* is a 1-form $A$ such that $B = dA$. We also use $A$ as the potential for $F$, $F = dA$.

When is a closed form exact? What are the conditions such that $Z^1(M) = B^1(M)$? Under these conditions then a form can be written as the differential of some potential and vanishes under a second application of the exterior derivative.

**Definition 1.126.** Let $S$ be a manifold. A (smooth) *path* in $S$ is a smooth map $\gamma : [0,T] \to S$. We can *integrate a* 1*-form over the path*, $E \in \Omega^1(S)$, in a natural way: if $\gamma$ is a path, $\gamma'(t)$ is a tangent vector at $\gamma(t)$, and then $E_{\gamma(t)}$ is a cotangent vector at $\gamma(t)$. Thus we define

$$\int_\gamma E = \int_0^T E_{\gamma(t)}(\gamma'(t))dt \tag{123}$$

**Definition 1.127.** If there is a path between any two points in $S$, $S$ is (path) *connected*. If not, a maximal connected subset is called a *connected component*.

In general we assume we are on a connected manifold, but if not, lots of our theorems apply with the caveat that you restrict yourself to specific connected components of the manifold.

**Definition 1.128.** Let $\gamma_0, \gamma_1 : [0,T] \to S$ be two paths from $p$ to $q$ in $S$. Then $\gamma_0$ and $\gamma_1$ are *homotopic* if there exists a $\gamma : [0,1] \times [0,T] \to S$ such that $\gamma(s, \cdot)$ is a path from $p$ to $q$ for any $s \in [0,1]$, and $\gamma(0,t) = \gamma_0(t)$ and $\gamma(1,t) = \gamma_1(t)$. The function $\gamma$ is called a *homotopy* between $\gamma_0$ and $\gamma_1$.

**Definition 1.129.** Given a manifold $S$, it is *simply connected* if any two paths between any two (fixed) points are homotopic.

**Theorem 1.130.** *The integral of a closed* 1*-form is the same over homotopic paths. That is if* $\gamma_0 \sim \gamma_1$ *and* $dE = 0$

$$\int_{\gamma_0} E = \int_{\gamma_1} E \tag{124}$$

*Proof.* Suppose $\gamma_0$ and $\gamma_1$ are paths from $p$ to $q$ which are homotopic. Then there exists a $\gamma : [0,1] \times [0,T] \to S$ such that $\gamma(s,0) = p$ and $\gamma(s,T) = q$ for any $s \in [0,1]$ and $\gamma(0,t) = \gamma_0(t)$ and $\gamma(1,t) = \gamma_1(t)$. Define

$$I(s) = \int_0^T E_{\gamma(s,t)}(\gamma'(s,t))\, dt$$

Working on a patch, we can switch to local coordinates, say $\{e_\mu\}$ with dual basis $\{f^\mu\}$, so that we can expand in the patch as

$$\begin{aligned} E_{\gamma(s,t)} &= E_\mu(\gamma(s,t)) f^\mu \\ \gamma'(s,t) &= [\gamma'(s,t)]^\nu e_\nu \end{aligned}$$

which we can combine into

$$\begin{aligned} E_{\gamma(s,t)}(\gamma'(s,t)) &= E_\mu(\gamma(s,t))[\gamma'(s,t)]^\nu f^\mu(e_\nu) \\ &= E_\mu(\gamma(s,t))[\gamma'(s,t)]^\mu \\ &= E_\mu(\gamma(s,t))\partial_t \gamma^\mu(s,t) \end{aligned}$$

Now we can compute

$$\begin{aligned} \partial_s I_s &= \int \partial_s[E_\mu(\gamma(s,t)\partial_t\gamma^\mu(s,t)]\, dt \\ &= \int [\partial_s E_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t) + E^\mu(\gamma(s,t))\partial_s\partial_t\gamma^\mu(s,t)]\, dt \\ &= \int [\partial_s E_\mu(\gamma(s,t))\partial_t\gamma^\mu(s,t) - \partial_t E_\mu(\gamma(s,t))\partial_s\gamma^\mu(s,t)]\, dt \\ &= \int \partial_\nu E_\mu(\gamma(s,t))[\partial_s\gamma^\nu\partial_t\gamma^\mu - \partial_t\gamma^\nu\partial_s\gamma^\mu]\, dt \end{aligned}$$

Noting that: $dE = (\partial_\mu E_\nu - \partial_\nu E_\mu)dx^\mu dx^\nu$ we get

$$\partial_s I_s = \int (dE)_{\mu\nu}\partial_s\gamma^\mu\partial_t\gamma^\nu\, dt$$

so $\partial_s I_s = 0$ if $dE = 0$, and so we have our result. □

**Theorem 1.131.** *Let $S$ be simply connected and $E$ a closed 1-form on $S$, then $E$ is exact.*

*Proof.* Pick and fix any $p \in S$. For any $q \in S$ define

$$\phi(q) = -\int_\gamma E$$

where $\gamma$ is a path from $p$ to $q$. Since $S$ is simply connected and $E$ is closed, by the previous theorem, this function is well-defined. We will show $E = -d\phi$, so $\phi$ is a scalar potential for $E$, and so $E$ is exact ($E = d(-\phi)$).

Consider any $v_q \in T_qS$, then $-d\phi(v_q) = -v_q(\phi)$. Pick a path $\gamma : [0, 2] \to S$ with $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma'(1) = v_q$. Then

$$\begin{aligned} E(v_q) &= E(\gamma'(1)) \\ &= \frac{d}{dt}\int_0^t E(\gamma'(s))\, ds\Big|_{t=1} \\ &= -\frac{d}{dt}\phi(\gamma(t))\Big|_{t=1} \\ &= -v_q(\phi) \\ &= -d\phi(v_q) \end{aligned}$$

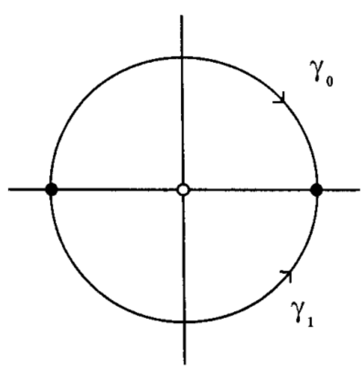
Hence, for any $v_q \in T_qS$ we have $E(v_q) = -d\phi(v_q)$ so $E = -d\phi$. □

**Proposition 1.132.** $\mathbb{R}^n$ *is simply connected.*

*Proof.* Let $p, q \in \mathbb{R}^n$ arbitrary. Suppose $\gamma_0(t)$ and $\gamma_1(t)$ are paths from $p$ to $q$. Then let $\gamma(s, t) = \gamma_0(t)(1 - s) + \gamma_1(t)s$. Then $\gamma(s, t)$ is a homotopy. □

**Example 1.133.** $\mathbb{R}^2 - \{0\}$ is NOT simply connected. Consider the paths on $[0, 1] \to S$ given by

$$\begin{aligned} \gamma_0(t) &= \langle \cos(\pi(1 - t)), \sin(\pi(1 - t))\rangle \\ \gamma_1(t) &= \langle \cos(\pi(1 - t)), -\sin(\pi(1 - t))\rangle \end{aligned}$$



Consider the 1-form

$$E = \frac{xdy - ydx}{x^2 + y^2}$$

Let $r = x^2 + y^2$, then

$$\begin{aligned}
E &= \frac{x}{r}dy - \frac{y}{r}dx \\
dE &= (\partial_x E) \wedge dx + (\partial_y E) \wedge dy \\
&= \left[\frac{y^2 - x^2}{r^2}dy - \frac{-2xy}{r^2}dx\right] \wedge dx + \left[-\frac{-2xy}{r^2}dy - \frac{x^2 - y^2}{r^2}dx\right] \wedge dx \\
&= 0
\end{aligned}$$

so $E$ is closed.

Now we compute

$$\begin{aligned}
\int_{\gamma_0} E_0 &= \int_0^1 E_{\gamma_0(t)}(\gamma_0'(t))\, dt \\
&= \int_0^1 E_\mu(\gamma_0(t))\partial_t(\gamma_0^\mu(t)))\, dt \\
&= \int_0^1 \left[E_x(\gamma_0)\gamma_0^{\prime,x}(t) + E_y(\gamma_0)\gamma_0^{\prime,y}(t)\right]\, dt \\
&= \int_0^1 [-\sin(\pi(1-t))\pi\sin(\pi(1-t)) - \cos(\pi(1-t))\pi\cos(\pi(1-t))] \\
&= -\pi
\end{aligned}$$

Similarly, the opposite integral is $\pi$. So $\int_{\gamma_0} E \neq \int_{\gamma_1} E$ for our closed form, so $\gamma_0$ and $\gamma_1$ must not be homotopic, so $\mathbb{R}^2 - \{0\}$ is not simply connected.

**Definition 1.134.** A path $\gamma : [0, T] \to S$ is a *loop* if it ends where it starts. If $\gamma(0) = \gamma(T) = p$ we say the loop is *based at* $p$ or $p$ is the *basepoint*. The loop is *contractible* if it is homotopic to a constant loop based at $p$, $\eta_p(t) \cong p$ for all $t$.

**Note 1.135.** On a simply connected manifold, all loops are contractible.

**Theorem 1.136.** *Let $E$ be a 1-form on any manifold $S$.*

1. *$E$ is closed iff $\int_\gamma E = 0$ for all contractible loops, $\gamma$.*
2. *$E$ is exact iff $\int_\gamma E = 0$ for all loops, $\gamma$.*

*Proof.* We proceed by construction in each case.

1. For part 1.
   - $\Rightarrow$ Suppose $E$ is closed and let $\gamma$ be any contractible loop based at $p$. Then it is homotopic to $\eta_p$, so by our previous theorem $\int_\gamma E = \int_{\eta_p} E = 0$.
   - $\Leftarrow$ Assume $\int_\gamma E = 0$ for all contractible loops $\gamma$ based at any $p$. Pick a chart giving coordinates $\{x^\mu\}$ about $p \in S$. Then consider the integral of $E$ around a tiny square of width/height $\epsilon$ in the $x^\mu - x^\nu$ plane. Then by Green's theorem

$$\int_\gamma E = \int_0^\epsilon \int_0^\epsilon (\partial_\mu E_\nu - \partial_\nu E_\mu)\, dx^\mu dx^\nu$$

using $\int_\gamma E = 0$ on the LHS and letting $\epsilon \to 0$ we have

$$0 = \epsilon^2 (\partial_\mu E_\nu - \partial_\nu E_\mu)|_p = \epsilon^2 (dE)_{\mu\nu}|_p$$

so $dE = 0$ if $\int_\gamma E = 0$ for all contractible loops.

2. For part 2.
   - $\Rightarrow$ Suppose $E$ is exact, then $E = d\phi$ for some 0-form $\phi$. Let $\gamma$ be any loop based at $p_0$

$$\begin{aligned}\int_\gamma E &= \int_0^T E_{\gamma(t)}(\gamma'(t))\, dt \\ &= \int_0^T (d\phi)_{\gamma(t)}(\gamma'(t))\, dt \\ &= \int_0^T [\gamma'(t)](\phi)|_{\gamma(t)}\, dt \\ &= \int_0^T \left[\tfrac{d}{ds}\phi(\gamma(s))\right]|_{s=t}\, dt \\ &= \int_0^T [\phi(\gamma(t))]'\, dt \\ &= \phi(\gamma(t)) - \phi(\gamma(0)) \\ &= 0\end{aligned}$$

   - $\Leftarrow$ Suppose $\int_\gamma E = 0$ for all loops $\gamma$. Then $E$ is at least closed. Suppose, for contradiction, that $E$ is NOT exact, then $S$ is not simply connected. Then we may choose two points $p, q \in S$ such that there exist paths $\gamma_1$ and $\gamma_2$ from $p$ to $q$, such that $\int_{\gamma_1} E \neq \int_{\gamma_2} E$. Now let $\gamma = \gamma_1 \cup \gamma_2^*$, i.e. $\gamma_1$ forward then $\gamma_2^*$ in reverse. So $\gamma$ is a loop based at $p$, so $\int_\gamma E = 0$ by hypothesis, but $\int_\gamma E = \int_{\gamma_1} E - \int_{\gamma_2} E \neq 0$, a contradiction.

□

**Example 1.137.** For any manifold $M$, $\mathbb{S}^1 \times M$ is NOT simply connected. Choose the local coordinates $\{x^\mu\}$, $\mu = 0, 1, \ldots, m$, where $m = \dim M$, where $x^0$ is the $\mathbb{S}^1$ coordinate with 0 and $2\pi$ identified. Consider the 1-form $\omega = (1, 0, 0, \ldots, 0)$, then $d\omega = 0$, but if we try to construct a 0-form $\phi$ such that $\omega = d\phi$ by $\phi(q) = \phi(0) + \int_\gamma \omega = \phi(0) + x^0$, we see for any path $\gamma$ fixed in $M$ and travelling around $\mathbb{S}^1$ we have multiple values for $\phi$ based on if $\gamma$ wraps around once, twice, ..., etc. i.e. $\phi$ is a multi-function, not a 0-form.

### 1.6.2 Stokes' Theorem

**Definition 1.138.** In $\mathbb{R}^n$ with coordinates $\{x^\mu\}$ the *closed half-space* is

$$\mathbb{H}^n = \{(x^1, \ldots, x^n) : x^n \geq 0\} \tag{125}$$

**Definition 1.139.** A function on $\mathbb{H}^n$ is *smooth* if it extends to a smooth function on the manifold

$$\{(x^1, \ldots, x^n) : x^n + \epsilon > 0\} \qquad \text{for some } \epsilon > 0 \tag{126}$$

**Definition 1.140.** An $n$-dimensional *manifold with boundary* is a (Hausdorff paracompact) topological space $M$ equipped with charts of the form:

$$\varphi_\alpha : U_\alpha \to \mathbb{R}^n \qquad \text{or} \qquad \varphi_\alpha : U_\alpha \to \mathbb{H}^n \tag{127}$$

where $U_\alpha$ are open sets covering $M$ such that the transition function $\varphi_\alpha \circ \varphi_\beta^{-1}$ are smooth where defined.

**Definition 1.141.** If $M$ is an $n$-dimensional manifold with boundary, the *boundary* $\partial M$ is the set of $p \in M$ such that some chart $\varphi_\alpha : U_\alpha \to \mathbb{H}^n$ maps $p$ to a point in

$$\partial\mathbb{H}^n = \{(x^1, \ldots, x^n) : x^n = 0\} \tag{128}$$

**Definition 1.142.** A function on a manifold with boundary is *smooth* if $f \circ \varphi_\alpha$ is smooth as a function on $\mathbb{R}^n$ or $\mathbb{H}^n$ for any chart $\varphi_\alpha$.

**Note 1.143.** The tangent space to a point on the boundary is still an $n$-dimensional vector space. The coordinates $\{x^1, \ldots, x^{n-1}\}$ are still as before, and $x^n = 0$ causes no issues because functions are required to extend to smooth functions on $-\epsilon < x^n < 0$.

**Definition 1.144.** Let $\omega$ be any $n$-form on $\mathbb{R}^n$. Let $\{x^\mu\}$ a coordinate system on $\mathbb{R}^n$. Then we define the *integral of the $n$-form*, $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$, as:

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f\, dx^1 \cdots dx^n \tag{129}$$

*Proof.* This map is well-defined. That is, if $\{x'^\mu\}$ is any other coordinate system then: $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$. We know if $T$ is the map such that:

$$dx'^\mu = T^\mu_\nu dx^\nu$$

where

$$T^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

i.e. the Jacobian. Thus we have that $dx'^1 \wedge \cdots \wedge dx'^n = (\det T) dx^1 \wedge \cdots \wedge dx^n$ means $f = f'(\det T)$. So we have well-definedness by the Jacobian change of variables formula:

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f\, dx^1 \cdots dx^n = \int_{\mathbb{R}^n} f'\, (\det T) dx^1 \cdots dx^n = \int_{\mathbb{R}^n} f'\, dx'^1 \cdots dx'^n = \int_{\mathbb{R}^n} \omega$$

so the integral of $\omega$ is coordinate-independent. $\square$

**Result 1.145.** *Let $M$ be an $n$-dimensional manifold with boundary (Hausdorff and paracompact). Let $\{\varphi_\alpha\}$ an atlas of charts for $M$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ or $\varphi_\alpha : U_\alpha \to \mathbb{H}^n$. Then we may always find a collection of smooth functions $\{f_\alpha\}$ on $M$ called partitions of unity such that:*

1. *$f_\alpha$ is zero outside $U_\alpha$.*
2. *Any point $p \in M$ has an open set containing it on which only finitely many of the $f_\alpha$ are non-zero.*
3. *For any $p \in M$, $\sum_\alpha f_\alpha(p) = 1$.*

**Note 1.146.** This allows us to basically stitch together integrals over charts by weighting our form against an $f_\alpha$, ensuring each point only gets counted with total weight 1 when integrating. That is, we decompose

$$\omega = \sum_\alpha f_\alpha\, \omega \tag{130}$$

so that $f_\alpha\, \omega$ vanishes outside $U_\alpha$. We thus have, in the $U_\alpha$ coordinates,

$$f_\alpha\, \omega = g_\alpha(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n \tag{131}$$

where $g_\alpha$ vanishes outside $U_\alpha$.

**Definition 1.147.** We define the *integral of* $\omega \in \Omega^n(M)$ as

$$\int_M \omega = \sum_\alpha \int g_\alpha(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n \tag{132}$$

where $f_\alpha \omega = g_\alpha(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$ and $f_\alpha$ is a partition of unity.

**Note 1.148.** Integration is independent of the charts and partition of unity.

**Proposition 1.149.** *Let $M$ be an oriented manifold with boundary. Then $\partial M$ is an oriented manifold in a natural way, with dimension $n-1$.*

*Proof.* Take an atlas of charts for $M$, and only consider those of the form $\varphi_\alpha : U_\alpha \to \mathbb{H}^n$. Define $V_\alpha = U_\alpha \cap \partial M$, so $V_\alpha$ are open subsets of $\partial M$. Define $\psi_\alpha = \varphi_\alpha|_{V_\alpha}$, then $\psi_\alpha : V_\alpha \to \mathbb{R}^{n-1}$ is continuous with continuous inverse, and so $\psi_\alpha \circ \psi_\beta^{-1}$ are smooth (and orientation preserving!) It is oriented by the fact that diffeomorphisms (can) preserve orientation. □

**Result 1.150** (Stokes' Theorem)**.** *Let $M$ be a compact $n$-manifold with boundary, and let $\omega$ be an $(n-1)$-form on $M$. Then*

$$\int_M d\omega = \int_{\partial M} \omega \tag{133}$$

**Note 1.151.** We can drop $M$ compact if $M$ vanishes outside a compact set.

**Definition 1.152.** Given $S \subseteq M$, $S$ is a *$k$-dimensional submanifold* of $M$, if for any $p$ in $S$ there is an open set $U \subseteq M$, containing $p$, and a chart $\varphi : U \to \mathbb{R}^n$ such that

$$S \cap U = \varphi^{-1}\mathbb{R}^k \tag{134}$$

Or for a *$k$-dimensional submanifold with boundary*

$$S \cap U = \varphi^{-1}\mathbb{H}^k \tag{135}$$

**Definition 1.153.** If $N$ is a manifold, and $\phi : N \to M$ is a smooth map such that $\phi(N)$ is a submanifold of $M$, then $\phi$ is an *embedding* of $N$ in $M$.

**Proposition 1.154.** *Let $M$ be a manifold.*

1. *Any open subset of $M$ is a submanifold.*
2. *Any submanifold of $M$ is a manifold in a natural way.*

*Further suppose $M$ has boundary.*

3. *If $S \subseteq M$ is a $k$-dimensional submanifold with boundary, then $S$ is a manifold with boundary in a natural way.*
4. *$\partial S$ is a $(k-1)$-dimensional submanifold of $M$.*

*Proof.* We proceed directly.

1. Let $U \subseteq M$ open. $U$ is obviously coverable by charts of $M$ restricted to $U$. The transition functions are smooth on $U$ since they are smooth on $M$. If not, $M$ would not be a manifold anyway, for we could just add $U$ to any chart for $M$.
2. Suppose $S \subseteq M$ is a submanifold of $M$ of dimension $k$, then there are sets $V_\alpha = S \cap U_\alpha = \varphi_\alpha^{-1}\mathbb{R}^k$, which cover $S$ and are open in the natural subset topology inherited from $M$ to $S$. As for charts, we choose $\psi_\alpha : V_\alpha \to \mathbb{R}^k$ by $\psi_\alpha = \varphi_\alpha|_{U_\alpha}$.
3. Exactly as above, but we also have $\psi_\alpha : V_\alpha = S \cap U_\alpha = \varphi_\alpha^{-1}\mathbb{H}^k$ mapping to $\mathbb{H}^k$, by $\psi_\alpha = \varphi_\alpha|_{U_\alpha}$ with appropriate projection.
4. Suppose $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ or $\mathbb{H}^n$ are the charts of the manifold of $M$, and $S$ is a $k$-dimensional submanifold with boundary, covered by the charts $\psi_\alpha : V_\alpha = S \cap U_\alpha \to \mathbb{R}^k$ or $\mathbb{H}^k$ by $\psi_\alpha = \varphi_\alpha|_{U_\alpha}$. The boundary $\partial S$ is the set of $s \in S$ such that some $\psi_\alpha : V_\alpha \to \mathbb{H}^k$ map $p$ to a point in $\partial\mathbb{H}^k$. So in the same way we chose $V_\alpha = \varphi_\alpha^{-1}\mathbb{R}^k$ we choose the $W_\alpha = \psi_\alpha^{-1}\partial\mathbb{H}^k$, with all appropriate projections.

□

### 1.6.3 deRham Cohomology

**Definition 1.155.** The $p^{th}$ *deRham cohomology group of $M$* is the vector space

$$H^p(M) = \frac{Z^p(M)}{B^p(M)} \tag{136}$$

An element of $H^p(M)$ is an equivalence class, where two closed $p$-forms $\omega, \omega' \in Z^p(M)$ are equivalent if

$$\omega - \omega' = d\mu \tag{137}$$

for some $(p-1)$-form, $\mu$. In this case, $\omega$ and $\omega'$ are *cohomologous* and lie in the same *cohomology class.*

**Note 1.156.** Intuitively, $\dim(H^p(M))$ is the number of "$p$-holes," i.e. objects preventing homotopies for $p$-surfaces. For example a 1-hole stops a curve homotopy like in $\mathbb{R}^2 - \{0\}$.

<u>$H^0(M)$</u>: A 0-form is a function $f$. It is closed if $0 = df = \partial_\mu f dx^\mu$. That is, if it is *locally constant*. The most general locally constant function is one that take constant value $c_i$ on the $M_i$ component of a manifold $M$.

$$H^0(M) \simeq \{\text{space of locally constant functions on } M\} \tag{138}$$
$$\dim(H^0(M)) = \{\text{number of connected components of } M\} \tag{139}$$
$$H^0(M) = \mathbb{R} \quad \text{iff } M \text{ is connected} \tag{140}$$

<u>$H^1(M)$</u>: $H^1(M) = \{0\}$ if $M$ is simply connected. More arbitrarily, if we find a set of $d$ closed 1-forms $\omega^1, \dots, \omega^d$ on $M$ such that no (non-trivial) linear combination is exact, then $H^1(M)$ is at least $d$-dimensional.

We can show a closed 1-form is not exact by Stokes' theorem. Consider $S^1 \subseteq M$ an embedded circle. If $\omega = df$ for some $f \in \Omega^0(M)$ then $\int_S \omega = \int_S df = \omega_{\partial S} f = 0$ since $\partial S = \emptyset$. So if we find a circle such that $\int_S \omega \neq 0$ then $\omega$ is *not* exact.

**Theorem 1.157.** *Suppose $\omega = d\mu \in B^p(M)$ is an exact $p$-form on $M$. Then for every compact $p$-dimensional manifold $S$ and map $\phi : S \to M$*

$$\int_S \phi^*\omega = \int_S \phi^* d\mu = \int_S d(\phi^*\mu) = \int_{\partial S} \phi^*\mu = 0 \tag{141}$$

**Corollary 1.158.** *If $S \subseteq M$ any compact orientable submanifold then*

$$\int_S \omega = 0 \tag{142}$$

*This occurs for $\phi = id$ in the previous theorem.*

**Result 1.159.** *The converse is also true: if $\int_S \phi^*\omega = 0$ for every map $\phi : S \to M$ of a $p$-dimensional manifold $S$ to $M$, then $\omega$ is exact.*

**Note 1.160.** Combining these two results: Exact forms are those that integrate to 0, always.

**Proposition 1.161.** *The pullback of a closed form is closed. The pullback of an exact form is exact.*

*Proof.* Suppose $\omega$ is closed, then $d(\phi^*\omega) = \phi^*(d\omega) = \phi^*(0) = 0$. Suppose $\omega$ is exact, $\omega = dA$, then $\phi^*\omega = \phi^*(dA) = d(\phi^*A)$. □

**Proposition 1.162.** *Given any map $\phi : M \to N$ there is a linear map $\phi^* : H^p(N) \to H^p(M)$, by*

$$\phi^*([\omega]) = [\phi^*\omega] \tag{143}$$

*for all $\omega \in Z^p(M)$. Further, if $\psi : N \to S$, then $(\psi\phi)^* = \phi^*\psi^*$.*

*Proof.* Since $\phi^*$ preserves closed/exact forms it preserves linear combinations, and so it preserves the linear combination: $\omega - \omega' = d\mu$ by $\phi^*\omega - \phi^*\omega' = d(\phi^*\mu)$. Linearity is built in, so we see it is well-defined, linear, and composition is trivial:

$$(\psi\phi)^*[\omega] = [(\psi\phi)^*\omega] = [\phi^*\psi^*\omega] = \phi^*\psi^*[\omega]$$

□

### 1.6.4 Gauge Freedom

**Definition 1.163.** Given a potential $A$ for $B$, that is $B = dA$, we can add any differential to $A$ without changing $B$. $A \to A + df$ is called a *gauge transform* and $A$ is said to have *gauge freedom*. Picking and $f$ is called *choosing the gauge*.

**Example 1.164.** Suppose we are working on $M = \mathbb{R} \times S$ with metric $dt^2 - {}^3g$. If the 1-form $A$ satisfies

$$A(\partial_t) = 0 \tag{144}$$

on $\mathbb{R} \times S$. In this case, we say we are in *temporal gauge*.

**Proposition 1.165.** *Given any exact 2-form on $M = \mathbb{R} \times S$, and $F = dA$, we can always choose $A$ so that it is in temporal gauge.*

*Proof.* $A = A_0 dt + A_S$ for some $A_0$ and $A_S$ on $S$. Define $f$ on $M$ by

$$f(t,p) = \int_0^t A_0(s,p)\,ds$$

and then let $A' = A - df$. Then $dA' = F$, and

$$\begin{aligned} A'(\partial_t) &= A_0(t,p) - (df(\partial_t))(t,p) \\ &= A_0(t,p) - (\partial_t f)(t,p) \\ &= A_0(t,p) - \partial_t \int_0^t A_0(s,p)\,ds \\ &= 0 \end{aligned}$$

so $A'$ is in temporal gauge. □

# 2 Gauge Fields

## 2.1 Symmetry

> "*You probably already know what a Lie group is...*"
>
> William Fulton and Joe Harris, *Representation Theory*, 2004

### 2.1.1 Lie Groups

**Definition 2.1.** The *general linear group* $GL(n, \mathbb{F})$, is the set of all $n \times n$ invertible matrices over $\mathbb{F}$. The *special linear group* is the subset with the property that any element has determinant 1. The *orthogonal group*, $O(p, q)$, is the set of $n \times n$ matrices which preserve an inner product on $\mathbb{R}^n$ with signature $(p, q)$. The $\mathbb{C}$-analog to $O(n)$ is $U(n)$.

The *Lorentz group* is $SO(3, 1)$.

**Definition 2.2.** The *Poincaré group* is the group of symmetries of Minkowski space. The group of all diffeomorphisms preserving spacetime intervals.

**Result 2.3.** *Any element of the Poincaré group is a product of a translation, Lorentz transform, and possibly a parity or time reversal.*

**Definition 2.4.** Let $G$ be a group and suppose it is also a (smooth) manifold under $\cdot : G \times G \to G$ and $^{-1} : G \to G$. Then $G$ is a Lie group.

**Definition 2.5.** If $G$ and $H$ are Lie groups, and $H$ is a subgroup and submanifold of $G$, then $H$ is a *Lie subgroup* of $G$.

**Definition 2.6.** A *Lie group homomorphism*, $\rho : G \to H$, is a group homomorphism from $G$ to $H$ which is also a $C^\infty$-map.

**Theorem 2.7.** *$GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are open subsets of $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ respectively. $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are Lie groups (and thus so are their Lie subgroups).*

*Proof.* We will do just one example. First note that $\mathbb{R}^{n^2}$ is a manifold of dimension $n^2$ in its own right. Now consider $f : GL(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2} \to \mathbb{R}$ by $f(M) = \det(M)$, since $f$ is just polynomial in the entries of $M$ it is continuous. Now consider the open set $V = \mathbb{R} - \{0\}$, then $GL(n, \mathbb{R}) = f^{-1}(V)$, so $GL(n, \mathbb{R})$ is open and thus is a submanifold of $\mathbb{R}^{n^2}$ with usual topology. □

**Definition 2.8.** Given a Lie group, $G$, define its *identity component* $G_0$ as the connected component containing the identity.

**Proposition 2.9.** *$G_0$ is a subgroup of $G$, and a Lie group.*

*Proof.* Really, we just need to show $G_0$ is closed. Consider $L_h : G \to G$ by $L_h g = hg$. Then $L_h$ is continuous since multiplication on a Lie group is smooth. Now, since $G_0$ is connected there exists a $\gamma : [0,1] \to G_0$ such that $\gamma(0) = e$ and $\gamma(1) = g$. Since $L_h$ is continuous, $L_h\gamma$ is a path in $G_0$ and $L_h\gamma(0) = h$ and $L_h\gamma(1) = hg$. Now let $\alpha : [0,1] \to G_0$ a path in $G_0$ such that $\alpha(0) = e$ and $\alpha(1) = h$, then concatenation $\alpha \star L_h\gamma$ is a path in $G_0$ from $e$ to $hg$. So $G_0$ is closed. The other group properties are straightforward. □

**Proposition 2.10.** *Let $G$ be a connected Lie group and $U \subseteq G$ any neighbourhood of $e$, then $G = \langle U \rangle$.*

*Proof.* Let $H = \langle U \rangle$. For any $h \in H$, $L_h(U) \subseteq H$. Since $L_h$ is a homeomorphism, $L_h(U)$ is an open neighbourhood of $h$ and $L_h(U)$ is open in $H$. $H \subseteq \bigcup L_h(U)$ so $H$ is open. Let $g \in G - H$, and say $x \in L_g(U) \cap H$, then $x = gu \in H$ for some $u \in H$. By closure, $xu^{-1} = g \in H$, a contradiction, thus $L_g(U) \cap H = \emptyset$. Hence $G - H = \bigcup L_g(U)$ is open, so $H$ is closed. Since $H$ is clopen and non-empty in a connected space, $H = G$. □

**Example 2.11.** Every element of $O(3)$ is a rotation about an axis or a rotation about an axis plus a reflection through some plane. The former are in the identity component, that is $SO(3)$.

First, realize $T \in O(3)$ is a purely real element of $U(3)$ and so we can diagonalize any element $T$ with its orthogonal eigenvectors, say as

$$U^{-1}TU = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

and we know that each $\lambda_i = e^{i\theta_i}$. Furthermore, we know $\det(T) = \pm 1$ and that roots of the characteristic equation come in conjugate pairs, so we may continue WLOG

$$\sim \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

and say that the $\pm 1$ eigenvalue corresponds to the eigenvector $\vec{a}$. Then we see if it is $+1$, it is just an element of $O(2)$ about $\vec{a}$. If it is $-1$, it is a rotation about the axis $\vec{a}$ after reflection. The last bit is obvious by continuity of the determinant.

**Result 2.12.** *There is no path from* $\mathbb{I} \in SO(3,1)$ *to* $PT$ *in* $SO(3,1)$. *We call* $SO_0(3,1)$ *the connected lorentz group. More generally* $O(3,1)$ *has 4 components:*

1. *The identity component:* $SO_0(3,1)$*;* $\Lambda_{00}$ *positive and* $\det = +1$.
2. *The parity component:* $\Lambda_{00}$ *positive and* $\det = -1$.
3. *The time component:* $\Lambda_{00}$ *negative and* $\det = -1$.
4. *The time-parity component:* $\Lambda_{00}$ *negative and* $\det = +1$.

*So each element is a product of a proper orthochronous transform and a discrete transform* $\{\mathbb{I}, P, T, PT\}$.

**Definition 2.13.** We say a group $G$ *acts* on a vector space $V$ is there is a map $\rho : G \to GL(V)$ such that

$$\rho(gh)v = \rho(g)\rho(h)v \tag{145}$$

for any $v \in V$ and $g, h \in G$. We call $\rho$ a *representation* of $G$ on $V$.

**Definition 2.14.** Let $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ be representations of $G$. $\rho$ and $\rho'$ are *equivalent* if there is a bijective linear map $T : V \to V'$ such that $\rho'(g) \circ T = T \circ \rho(g)$ for all $g \in G$.

**Example 2.15.** If $G$ is a matrix group, $G \leq GL(n, \mathbb{F})$, then it naturally defines a representation, called the *fundamental representation*, on $V = \mathbb{F}^n$.

**Definition 2.16.** Let $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ be representations of $G$ on $V$ and $V'$. Then the *direct sum representation* $\rho \oplus \rho'$ is the rep $(\rho \oplus \rho') : G \to GL(V \oplus V')$ by

$$(\rho \oplus \rho')(g)(v, v') = (\rho(g)v, \rho'(g)v') \tag{146}$$

**Definition 2.17.** Let $V$ and $V'$ be vector spaces with bases $\{e_i\}$ and $\{e'_j\}$ respectively. Then the *tensor product* $V \otimes V'$ is the vector space with basis $\{e_i \oplus e'_j\}$. Given $v = v^i e_i \in V$ and $v' = v'^j e_j \in V'$, we say

$$v \otimes v' = v^i v'^j e_i \otimes e'_j \tag{147}$$

**Proposition 2.18.** *For any bilinear function* $f : V \times V' \to W$ *to some vector space* $W$*, there exists a unique linear function* $F : V \otimes V' \to W$ *such that*

$$f(v, v') = (F \circ \varphi)(v, v') = F(v \otimes v') \tag{148}$$

*so that the following diagram commutes*

$$\begin{array}{ccc} V \times V' & \xrightarrow{\varphi} & V \otimes V' \\ & \searrow^{f} & \downarrow F \\ & & W \end{array} \tag{149}$$

*Proof.* $f$ is bilinear so $f(v,v') = v^i v'^j f(e_i, e'_j)$, so we will define $F$ by $F(e_i \otimes e'_j) = f(e_i, e'_j)$. The basis $e_i \otimes e'_j$ are linearly independent, so the function is unique. □

**Definition 2.19.** Let $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ be representations of $G$ on $V$ and $V'$. Then the *tensor representation* $\rho \oplus \rho'$ is the rep $(\rho \otimes \rho') : G \to GL(V \otimes V')$ by

$$(\rho \otimes \rho')(g)(v \otimes v') = \rho(g)v \otimes \rho'(g)v' \tag{150}$$

**Definition 2.20.** Suppose $\rho : G \to GL(V)$ is a rep, and suppose $V \leq V'$ a vector subspace such that $\rho(g)v' \in V'$ for all $v' \in V'$ and $g \in G$. Then $V'$ is an *invariant subspace* of $V$. Furthermore, the emergent representation $\rho'$ of $G$ on $V$ by

$$\rho'(g)v' = \rho(g)v' \tag{151}$$

for all $v' \in V'$, is called a *subrepresentation.*

**Definition 2.21.** $\rho$ is *irreducible* if it has no non-trivial proper invariant subspaces.w

**Definition 2.22.** If $G$ is compact, every rep is equivalent to a direct sum of irreps.

**Theorem 2.23** (Schur's Lemma)**.** *If $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ are irreps of $G$ and $\varphi : V \to W$ is a $G$-module homomorphism (a homomorphism-equivalency of $\rho_V$ and $\rho_W$) then*

1. *$\varphi$ is an isomorphism, or $\varphi = 0$.*
2. *If $V = W$, then $\varphi = \lambda\,\mathbb{I}$ for some $\lambda \in \mathbb{C}$.*

*Proof.* This is one of my favourite theorems in representation theory.

1. Let $g \in G$ arbitrary, and $v \in \ker\varphi$ arbitrary, then $\varphi(\rho_V(g)v) = \rho_W(g)(\varphi v) = 0$, so $\rho_V(g)v \in \ker\varphi$. So $\ker\phi$ is an invariant subspace of $V$. Similarly, $\operatorname{im}\varphi$ is an invariant subspace of $W$. Thus we have $\ker\varphi = \{0\}$ or $V$ and $\operatorname{im}\varphi = W$ or $\{0\}$ respectively, giving the two cases.
2. If $V = W$ then since $\varphi$ must have an eigenvalue, $\lambda \in \mathbb{C}$, by algebraic closure of $\mathbb{C}$, then $\varphi - \lambda\,\mathbb{I}$ has a non-zero kernel which is atleast 1-dimensional. From proving 1 we know that the kernel of the map $\varphi - \lambda\,\mathbb{I}$ is an invariant subspace of $V$, thus $\varphi - \lambda\,\mathbb{I} = 0$ and we get $\varphi = \lambda\,\mathbb{I}$.

□

**Corollary 2.24.** *Every irrep of an abelian group is* 1*-dimensional.*

*Proof.* If $G$ is abelian, all $\rho(g)$ commute, so it must be that $\rho(g) = \lambda_g\,\mathbb{I}$ for all $g$, and so every subspace is invariant and $\rho$ is one-dimensional. □

**Definition 2.25.** For any rep $\rho$ of $G$ on a vector space $V$, there is a *dual* or *contragradient* representation $\rho^* : G \to GL(V^*)$ by

$$(\rho^*(g)w)(v) = w(\rho(g^{-1})v) \tag{152}$$

for $w \in V^*$. I.e. we have $\rho^*(g) = \rho(g^{-1})^T$.

**Definition 2.26.** A rep is *unitary* if $\rho(g)$ is unitary for all $g \in G$. A rep is *projective* if it holds up to a phase

$$\rho(1) = e^{i\theta} \tag{153}$$

$$\rho(g)\rho(h) = e^{i\theta(g,h)}\rho(gh) \tag{154}$$

the phase function $e^{i\theta(g,h)}$is called the *cocycle* of $g$ and $h$.

**Note 2.27.** Cocycles satisfy the cocycle condition

$$e^{i\theta(g,h)}e^{i\theta(gh,k)} = e^{i\theta(g,hk)}e^{i\theta(h,k)} \tag{155}$$

We can change any projective rep to an equivalent one by multiplication by a phase, in particular, we may choose $\rho(1) = \mathbb{I}$. If we cannot make $\theta(g,h)$ vanish for all $g, h \in G$, then the cocycle $e^{i\theta(g,h)}$ is called *essential.*

### 2.1.2 Lie Algebras

**Definition 2.28.** A *Lie algebra* $\mathfrak{g}$ is an algebra equipped with a bilinear skew-symmetric map satisfying the Jacobi identity:

1. $[v, w] = -[w, v]$
2. $[u, \alpha v + \beta w] = \alpha[u, v] + \beta[u, w]$
3. $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$

**Definition 2.29.** A *Lie algebra homomorphism* is a linear map $f : \mathfrak{g} \to \mathfrak{h}$ such that

$$f([v, w]) = [f(v), f(w)] \tag{156}$$

When $f$ is a bijection, it is an *isomorphism.*

**Note 2.30.** Given a Lie group (manifold) $G$, we defined the commutator on Vect($G$) by $[v, w](f) = v(w(f)) - w(v(f))$. This is an infinite dimensional Lie algebra

**Note 2.31.** The left multiplication map for any $g \in G$ is a diffeomorphism. Thus we may consider the pushforward of vector fields that it defines

$$(L_g)_* : \text{Vect}(G) \to \text{Vect}(G) \tag{157}$$

**Definition 2.32.** A vector field $v \in \text{Vect}(G)$ is *left-invariant* if $(L_g)_* v = v$ for all $g \in G$.

**Proposition 2.33.** *Let $M$ a manifold, $v, w \in \text{Vect}(M)$, and $\phi$ a diffeomorphism of $M$. Then*

$$\phi_*[v, w] = [\phi_* v, \phi_* w] \tag{158}$$

*Proof.* Using $N = M$ to denote the tangent space, the pushforward of a vector field on $M$ is defined pointwise by:

$$\begin{aligned} & (\phi_* v)_{\phi(p)} && = \phi_*(v_p) \\ \iff & (\phi_* v)_{\phi(p)}(f) && = [\phi_*(v_p)](f) \\ \iff & [\phi_* v] f(\phi(p)) && = v(\phi^* f)(p) \\ \iff & (\phi_* v)(f) \circ \phi && = v(\phi^* f) \end{aligned}$$

Since a vector field is determined by its points and $\phi : M \to M$ is a diffeomorphism

$$\begin{aligned} (\phi_*[v, w])_{\phi(p)}(f) &\equiv \phi_*([v, w]_p)(f) \\ &= [v, w]_p(\phi^* f) \\ &= v_p(w(\phi^* f)) - w_p(v(\phi^* f)) \\ &= v_p([phi_* w](f) \circ \phi) - w_p([\phi_* v](f) \circ \phi) \\ &= v_p(\phi^*([\phi_* w](f))) - w_p(\phi^*([\phi_* v](f))) \\ &= [\phi_* v_p]([\phi_* w](f)) - [\phi_* w_p]([\phi_* v](f)) \\ &= [\phi_* v]_{\phi(p)}([\phi_* w](f)) - [\phi_* w]_{\phi(p)}([\phi_* v](f)) \\ &= [\phi_* v, \phi_* w]_{\phi(p)}(f) \end{aligned}$$

Thus $\phi_*[v, w] = [\phi_* v, \phi_* w]$. □

**Corollary 2.34.** *The left-invariant vector fields form a subalgebra of* $\text{Vect}(G)$*.*

*Proof.* We see immediately that $\text{Vect}(G)^L$ is a vector subspace, since $(L_g)_*$ is linear. $\text{Vect}(G)^L$ is a subalgebra because it inherits the bracket and $L_g : G \to G$ is a diffeomorphism so

$$(L_g)_*[v, w] = [L_{g*} v, L_{g*} w] = [v, w]$$

□

**Theorem 2.35.** *Given any Lie group $G$, the subalgebra $\text{Vect}(G)^L \leq \text{Vect}(G)$ is isomorphic to $T_e G$. Hence we call $\mathfrak{g} = T_e G$ the Lie algebra of $G$ since it is the one naturally associated with $G$.*

*Proof.* Take any $v_e \in \mathfrak{g}$, that is, a tangent vector at $e \in G$, then define $v \in \text{Vect}(G)$ by translating this vector from the tangent space at the identity around, that is

$$v_g = (L_g)_* v_e$$

We see that such a $v$ is left-invariant, because for any $h \in G$, we have:

$$(L_g)_* v_h = (L_g)_*(L_h)_* v_e = (L_g L_h)_* v_e = (L_{gh})_* v_e = v_{gh} = v_{L_g(h)}$$

so that $(L_g)_* v = v$ by the definition of a pushforward of a vector field. So each element of $T_eG$ generates a vector field in $\text{Vect}(G)^L$.

Conversely, consider any $v \in \text{Vect}(G)^L$ and consider $v_e \in \mathfrak{g}$. So each element of $\text{Vect}(G)^L$ generates an element of $T_eG$. □

**Corollary 2.36.** *There is a natural Lie bracket on* $\mathfrak{g}$*, following from the isomorphism* $\text{Vect}(G)^L \cong \mathfrak{g}$.

**Result 2.37.** *Let* $G$ *be a Lie group,* $\mathfrak{g}$ *its associated Lie algebra, then there exists a smooth map*

$$\exp : \mathfrak{g} \to G \tag{159}$$

*satisfying:*

1. $\exp(0)$ *is the identity of* $G$.
2. $\exp(sX)\exp(tX) = \exp((s+t)X)$ *for all* $X \in \mathfrak{g}$ *and* $s, t \in \mathbb{R}$.
3. $\frac{d}{dt}(\exp(tX))|_{t=0} = X$

*Thus, by the inverse function theorem,* exp *maps any sufficiently small open neighbourhood of* $0 \in \mathfrak{g}$ *onto an open set containing* $e \in G$*, and generates the whole of* $G_0$.

**Corollary 2.38.** *Any element of the identity component of* $G$ *is the product of elements of the form* $\exp(X)$ *for* $X \in \mathfrak{g}$.

**Note 2.39.** For $G$ a matrix Lie group, $[\cdot,\cdot]$ is just the matrix commutator and exp is the matrix exponential.

**Theorem 2.40.** *For any homomorphism* $\rho : G \to H$*, the map* $d\rho = \rho_* : T_eG \to T_eH$ *is a homomorphism between the Lie algebras.*

*Proof.* First we note that $\rho_*$ really is a map $T_eG \to T_eH$ since it just pushes forward vectors $v_{e_G}$ to $\tilde{v}_{e_H}$. Now let $v, w \in \text{Vect}(G)$

$$\begin{aligned} d\rho([v,w]) &= \rho_*[v,w] \\ &= [\rho_* v, \rho_* w] \\ &= [d\rho(v), d\rho(w)] \end{aligned}$$

□

**Result 2.41.** *The following results sum up the Lie algebra/group correspondence:*

1. *$G$ determines a $\mathfrak{g}$ uniquely*
2. *$\mathfrak{g}$ determines a $G$ that is simply connected uniquely, and all other Lie groups with $\mathfrak{g}$ as an algebra are covered by $\mathfrak{g}$.*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(d\rho)_e} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\rho} & H \end{array}$$

**Definition 2.42.** A *Lie algebra representation* of a Lie algebra $\mathfrak{g}$ on a vector space $V$ is a Lie algebra homomorphism

$$f : \mathfrak{g} \to \mathfrak{gl}(V) \tag{160}$$

where $\mathfrak{gl}(V)$ is the Lie algebra of linear operators on $V$.

## 2.2 Bundles and Connections

Fields are sections of bundles, not maps $F : M \to V$, because those maps really only exist when we work locally in a chart. Connections allows us to compare vector fields at different parts of the manifold.

### 2.2.1 Bundles

**Definition 2.43.** A *bundle* is a triple $(E, M, \pi)$ consisting of a manifold $E$, called the *total space*, a manifold $M$, called the *base space*, and a *projection map* $\pi : E \to M$.

For each $p \in M$, the *fiber over* $p$ is $E_p = \pi^{-1}(p) = \{q \in E : \pi(q) = p\}$. $E$ is thus the union of its fibers over $M$, $E = \bigcup_{p\in M} E_p$, and we say $E$ is a *bundle over* $M$.

$M$ is typically a physical space/spacetime.

**Definition 2.44.** The *tangent bundle*, $TM$, of a manifold $M$ is simply

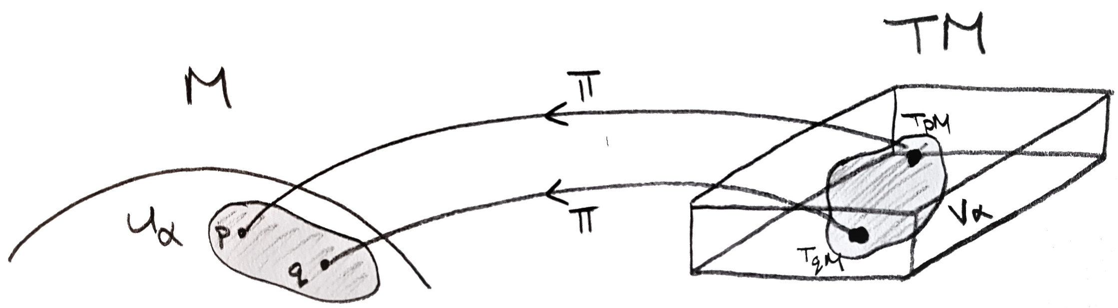
$$TM = \bigcup_{p\in M} T_pM \tag{161}$$

The projection map $\pi : TM \to M$ takes each $v_p \in T_pM$ to $p \in M$. Thus the fiber over $p$ is $T_pM$.

**Note 2.45.** If $M$ is an $n$-dimensional manifold it looks locally like $\mathbb{R}^n$. Since each $T_pM \cong \mathbb{R}^n$, specifying a point of $TM$ is picking a point $p \in M$ and a vector field $v$ evaluated at $p$. Thus locally $TM \sim \mathbb{R}^n \times \mathbb{R}^n$.

**Theorem 2.46.** *$TM$ is an $n^2$-dimensional smooth manifold.*

*Proof.* Given a manifold $M$ of dimension $n$ we have smooth charts $\varphi_\alpha : U_\alpha \subseteq M \to \mathbb{R}^n$ covering $M$. We define $V_\alpha$ to be all tangent vectors to $M$ associated to points in the chart $U_\alpha$, so $V_\alpha = \{v_p \in TM : \pi(v_p) \in U_\alpha\}$.




Note, in this picture, elements of $TM$ are not tangent planes, those are the fibers above the points $p$ and $q$. The actual elements of $TM$ are vectors.

Now we show the $V_\alpha$ cover $TM$. Let $v_p \in TM$ arbitrary, so $v_p \in T_pM$ for some $p \in M$ and thus $\pi(v_p) = p \in M$. Since $p \in M$ and $\{U_\alpha\}$ cover $M$, $\pi(v_p) \in U_\alpha$ for some $\alpha$, so $v_p \in V_\alpha$.

Now we define the map: $\psi_\alpha : V_\alpha \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\psi_\alpha(v_p) = (\varphi_\alpha(\pi(v_p)), (\varphi_\alpha)_* v_p)$$

The first part of the map takes the $v_p$ to $p$ and then to its corresponding point in the "first" $\mathbb{R}^n$, and the second part of the map pushes $v_p$ forward to a vector in $\mathbb{R}^n$.

Now we equip $TM$ with the topology generated by unions of sets $\mathcal{O}$, satisfying: $\mathcal{O} \subseteq V_\alpha$ and $\psi_\alpha(\mathcal{O}) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is open.

Then we see the $\psi_\alpha$ are smooth charts (and so $TM$ is smooth) since $\varphi_\alpha$ are smooth and $\psi_\alpha$ is just a function of $\varphi_\alpha$ and its pushforward $(\varphi_\alpha)_*$. The transition functions are just compositions of $\psi_\alpha$, which are then smooth by the $\varphi_\alpha$ argument again. $\pi$ is smooth since on a patch $\tilde{\pi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

1. Use $\psi_\alpha^{-1}$ to take $x \in \mathbb{R}^n \times \mathbb{R}^n$ back to $v_p$.
2. Take $v_p$ to $p$ with $\pi$.
3. Take $p$ to $\mathbb{R}^n$ by $\varphi_\alpha$

i.e. $\tilde{\pi} = \varphi_\alpha \circ \pi \circ \psi_\alpha^{-1}$. □

**Definition 2.47.** Given manifolds $M$ and $F$, the *trivial bundle* over $M$ with *standard fiber* $F$ is defined by

$$E = M \times F \qquad \pi(p, f) = f \tag{162}$$

for all $(p, f) \in E$. In this case, the fiber of $p$ is just a respective copy of $F$, i.e. $E_p = \{p\} \times F$.

**Definition 2.48.** Suppose $\pi : E \to M$ and $\pi' : E' \to M'$ are two bundles. A *bundle morphism* from the first to the second is a map $\psi : E \to E'$ and a map $\phi : M \to M'$ such that $\psi$ maps each fiber $E_p$ into the fiber $E'_{\phi(p)}$

**Proposition 2.49.** *Given bundles $\pi : E \to M$ and $\pi' : E' \to M'$, then $\psi : E \to E'$ and $\phi : M \to M'$ defines a bundle morphism if and only if $\pi' \circ \psi = \phi \circ \pi$. Thus $\psi$ uniquely determines $\phi$ and the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\phi} & M' \end{array} \tag{163}$$

*Proof.* □