The Logistic Map and Bifurcation Diagrams

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The Logistic Map

The logistic map is a recurrence relation given by

$$x_{j+1} \coloneqq r x_j (1 - x_j) \tag{1}$$

for $x_j \in [0, 1]$ and a fixed $r \in [1, 4]$.

Introduction

Interest in the logistic map stems from evaluating its long-term behaviour for different values of the *control parameter*, r. Despite its simple definition the logistic map is an ideal system for illustrating chaos, including

- Stable behaviour growing increasingly complex
- A clear transition to chaos at a point, r_∞
- Self-similar and fractal behaviours
- Islands of stability amidst chaotic behaviour

Stable and Unstable Behaviour

Below we iterate (1) for various values of r with initial parameter $x_1 = 2/3$. Different values of r lead to very different qualitative behaviour after the initial action of transient points is damped.

- A sequence of points $x_j, x_{j+1}, x_{j+2}, \ldots$ is an *orbit*.
- If an orbit alternates between m points then the orbit is an m-cycle of those *fixed points*.
- The value of r where a 2^n-cycle splits into a 2^{n+1}-cycle is a *bifurcation point*.

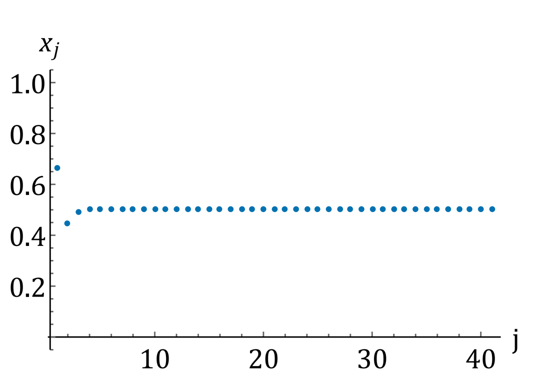

Fig. 1: $r = 2.0$ (fixed point)

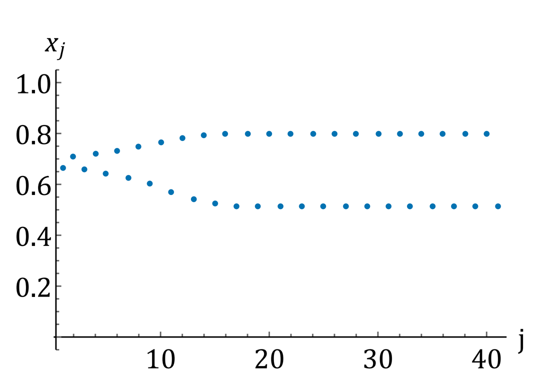

Fig. 2: $r = 3.2$ (2-cycle)

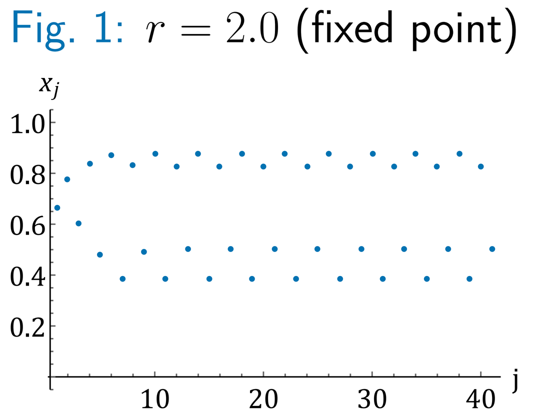

Fig. 3: $r = 3.5$ (4-cycle)

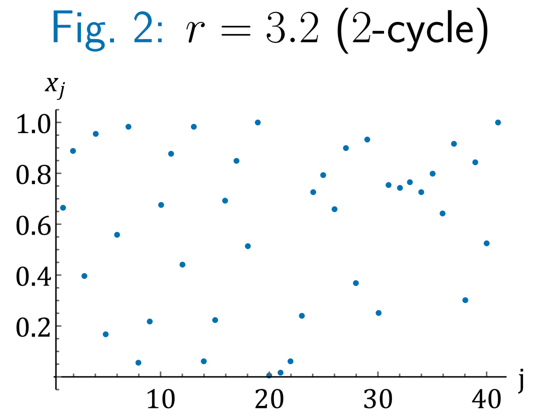

Fig. 4: $r = 4.0$ (chaos)

Bifurcation Diagrams

A *bifurcation diagram* shows the change in orbit structure as r varies by marking fixed points for each value of r. To find these points we iterate (1) thousands of times and then assume that all of the transient behaviour is removed so that further iterations only produce fixed points, we then plot the results.

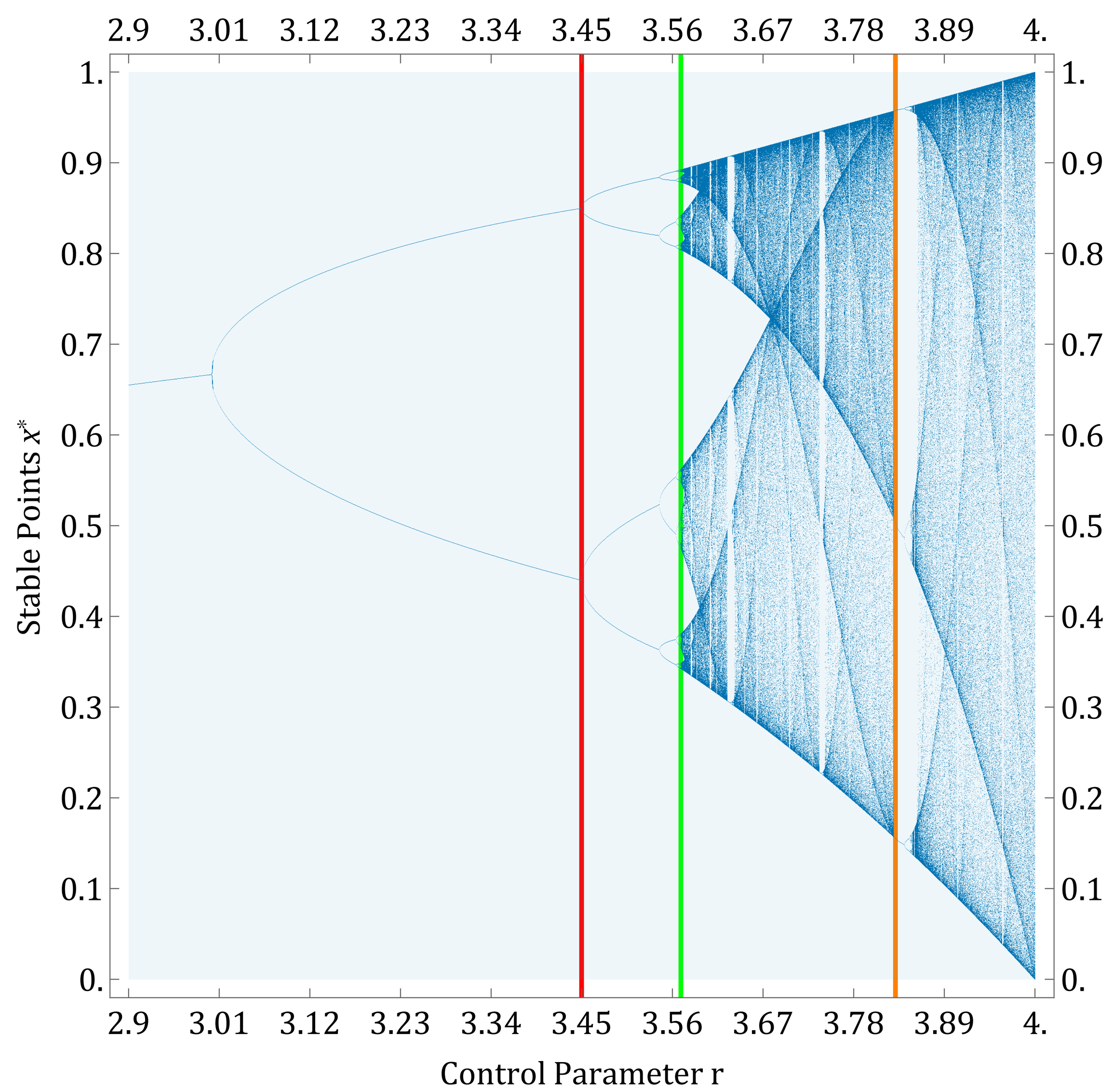

Fig. 5: A bifurcation diagram with initial value $x_1 = 2/3$. The bifurcation point r_2=3.4494, the *limit of chaos* r_∞=3.5699, and a large island of stability with no chaotic behaviour $1+\sqrt{8}$ are marked. The r-value of bifurcation points is independent of x_1.

For $r < r_\infty$ bifurcations are 'pitchfok-shaped.' By Taylor expansion around a fixed point, we see why:

Theorem *Let $f(x) = rx(1-x)$. If an m-cycle bifurcates to a $2m$-cycle at x_b, then $|f^{(2m)\prime}(x_b)| = 1$. If p is a fixed point of the m-cycle, then $|f^{(2m)\prime}(p)| > 1$ and p is unstable after bifurcation.*

The Lyapunov Exponent

The *Lyapunov exponent* is a measure of the speed of divergence of infinitesimally close orbits. For us

$$\lambda(r) \coloneqq \frac{1}{m}\sum_{i=1}^{m} \ln |f'(x_i^\star)| \tag{2}$$

$$\lambda(r) \coloneqq \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \ln |f'(x_i^\star)| \tag{3}$$

where $x_i^\star$ is the i-th element of an m-cycle. We use (2) for m-cycles, and (3) in the region of chaos.

- If a system is not sufficiently pathological, $\lambda(r) > 0 \implies$ chaos and $\lambda(r) < 0 \implies$ stability.
- For the logistic map, $\lambda(r) = 0$ at bifurcations.
- For $r < r_\infty$, $\lambda(r) \leq 0$. For $r \geq r_\infty$, $\lambda(r)$ takes both positive and negative values.

The most chaotic point occurs at the maximum $\lambda(4)$. John von Neumann suggested using this as a random number generator.

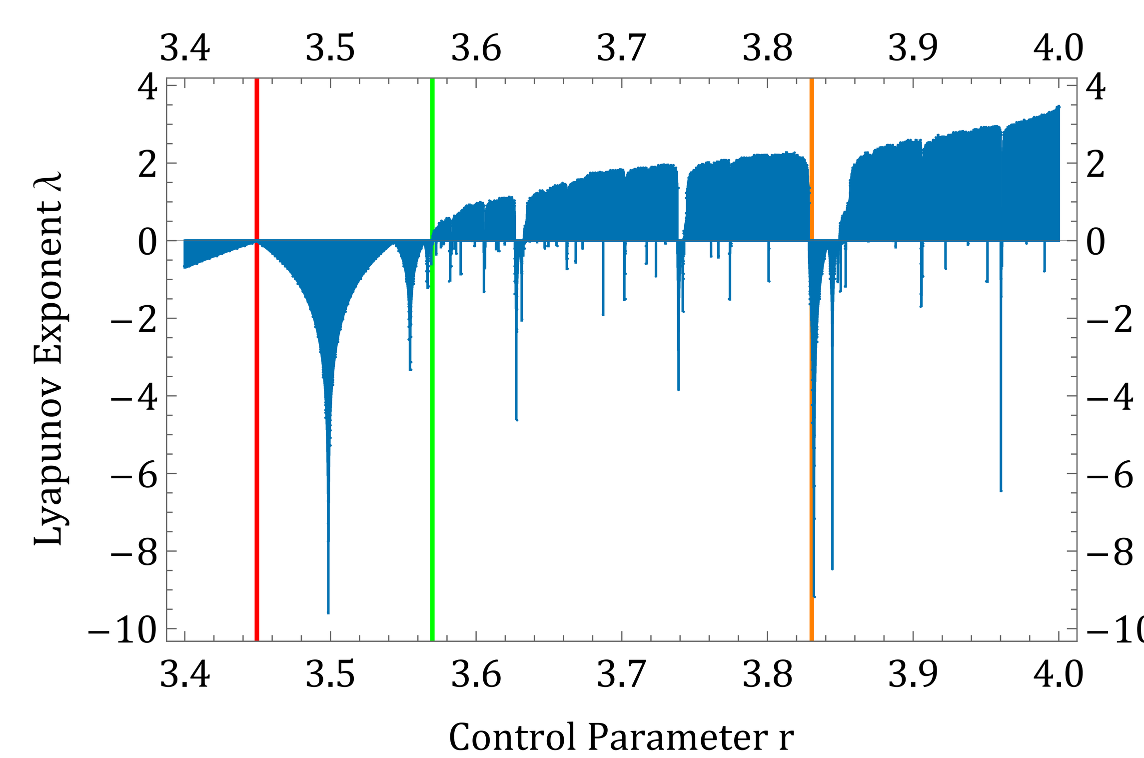

Fig. 6: A plot of the Lyapunov exponent $\lambda(r)$. The bifurcation point r_2=3.4494 is seen to be 0, a transition to chaos occurs as $\lambda(r) > 0$ at the limit of chaos r_∞=3.5699, and the island of stability at $1+\sqrt{8}$ is seen to dive deeply negative.

An Island of Stability

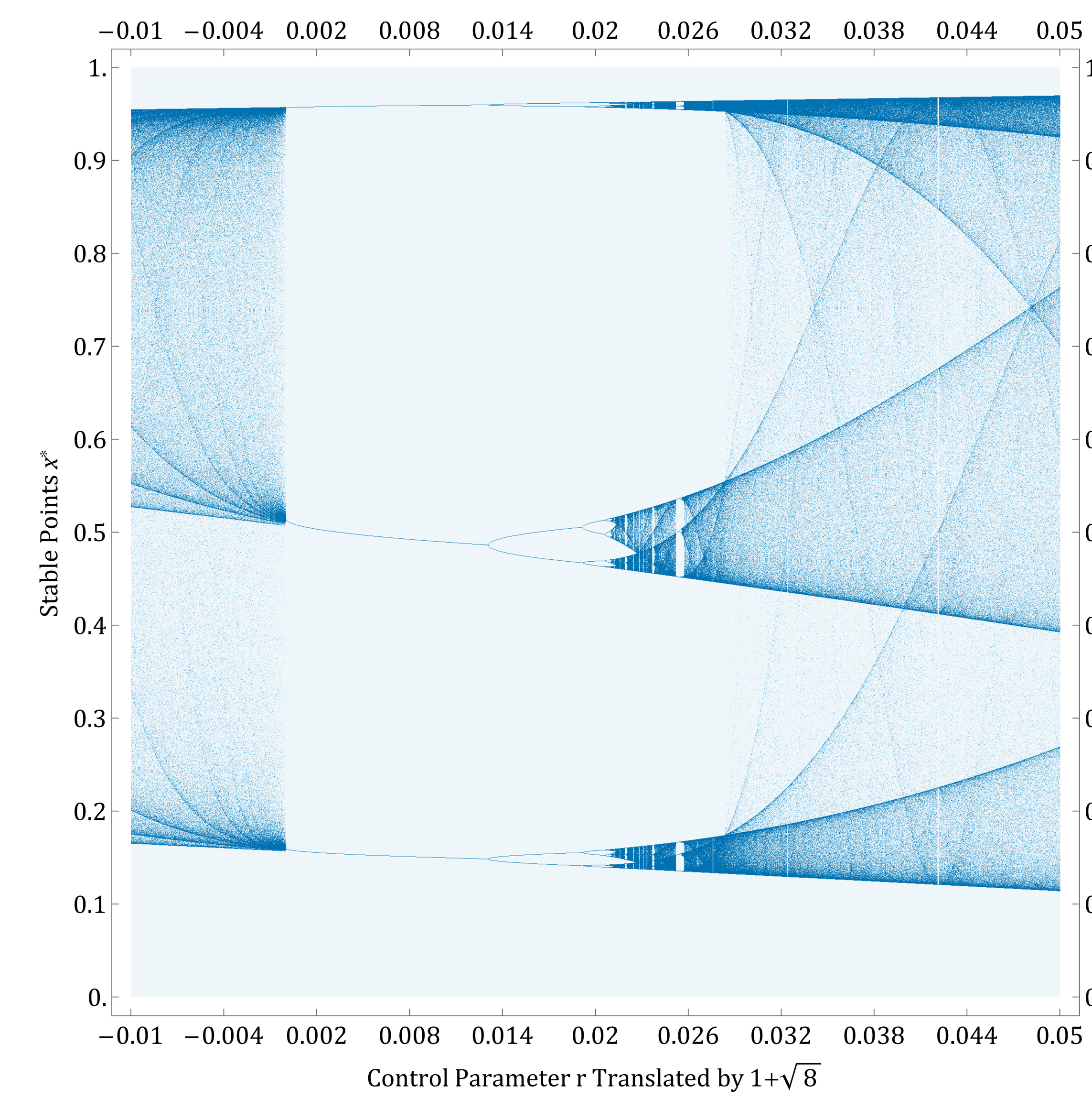

Fig. 7: The bifurcation digram from Fig. 5 zoomed into $1+\sqrt{8}$ (and translated). Chaos from $r < 1+\sqrt{8}$ closes off abruptly to a 3-cycle. The 3-cycles bifurcate, and the middle one shows very clear self-similarity to Fig. 5.

Using numerical methods we solve $f^{(3)}(x) = x$ and confirm the existence of a 3-cycle at $1+\sqrt{8}$. In particular, there are 3 double roots which are neutrally stable and bifurcate, and two unstable roots.

Finding Bifurcations: A Numerical Problem

Analytic solutions for bifurcation points become unruly quickly. Instead, we find bifurcation points in the same manner as we generate our diagram. If the bifurcation diagram is an m-cycle for some set of $r < r_\infty$, then the bifurcation point is the infimum of this set. The first 6 bifurcation points are approximately: $3, 3.4494, 3.5440, 3.5644, 3.5687, 3.5696$. Ratios of successive differences tend to *Feigenbaum's Constant*:

$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \to 4.669201\ldots \quad \text{as } n \to \infty$$

References

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