# Aperiodic Tilings

## An Introduction

Justin Kulp

October, 4th, 2017

# Background: Tiling

## Definition (Tiling)

A *tiling* of $\mathbb{R}^d$ is a non-empty countable collection of closed sets in $\mathbb{R}^d$, $\mathcal{T} = \{T_i : i \in I\}$, subject to the constraints that:

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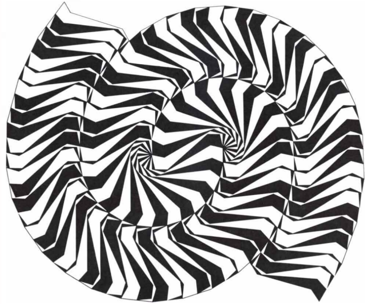
- The *symmetries* of $\mathcal{T}$ are isometries that map $\mathcal{T}$ onto itself, and $\mathcal{T}$ is *nonperiodic* if it has no translational symmetry.

# Background: Tiling

Periodic tiling by M.C. Escher

Nonperiodic tiling by Heinz Voderberg



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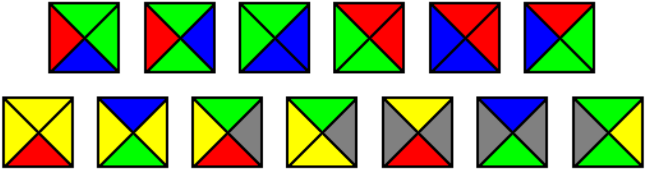

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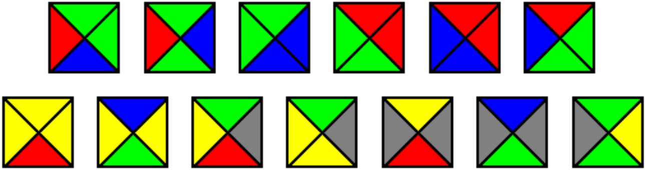

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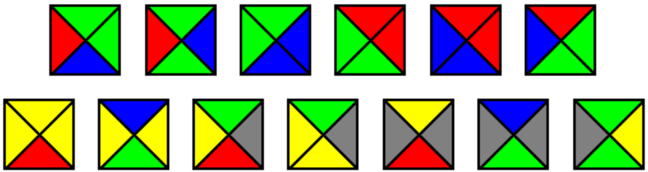

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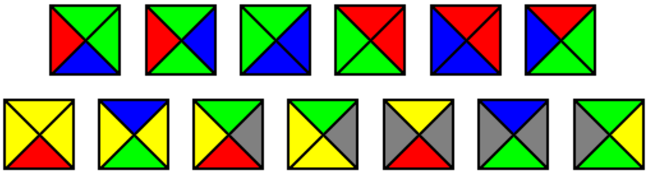

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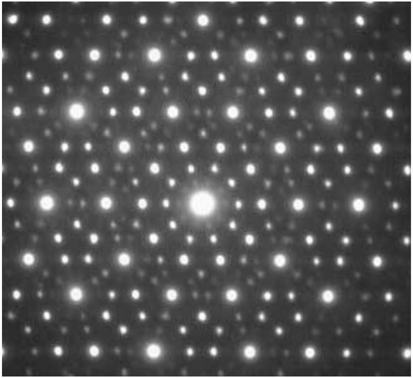
- A set of prototiles that only admits nonperiodic tilings is called *aperiodic*.

# Background: Quasicrystals

- In 1982 Dan Shechtman (2011 Nobel Prize, Chemistry) produced a sample of $Al_6Mn$ with the diffraction pattern



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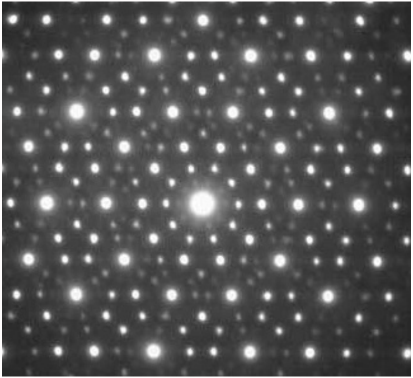

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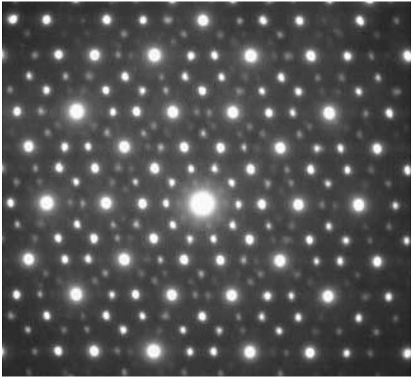

  - Classically forbidden diffraction pattern.
  - Explainable as the diffraction of a lattice described by a quasiperiodic function: $\sin(x) + \sin(\tau x)$.

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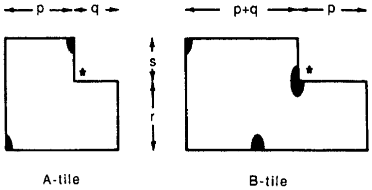


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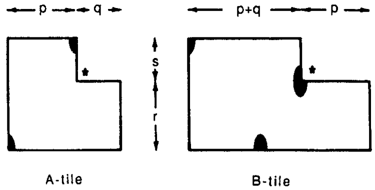
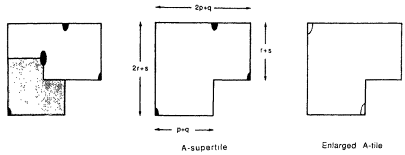



with the rule that black ellipses must be formed under composition.
- When we try to put the tiles together, we find there are only a limited number of ways to do it.

# Substitution Tilings: Ammann's A2 Tiles

- When we try to combine the $A$ and $B$ tile at the $\star$ we find they can only be assembled



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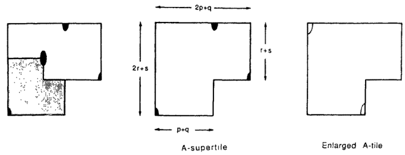
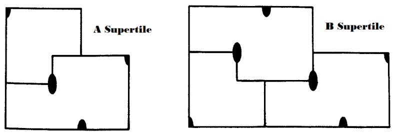



- If $p/q = r/s = \tau$ we find the matching rules allow us to UNIQUELY compose $A$ and $B$ *supertiles*, scaled up by a factor of $\tau$.



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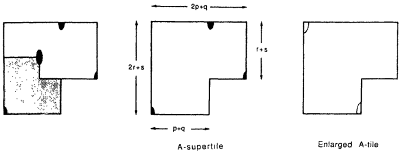


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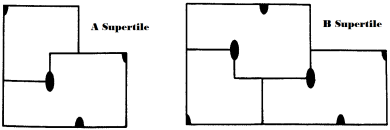


- We also find the inherited matching rules are the same amongst the supertiles as the regular tiles.

# Substitution Tilings: Aperiodicity Theorem

## Theorem (Substitution Tilings are Aperiodic)

*A set of prototiles, $\mathcal{P}$, is aperiodic if*

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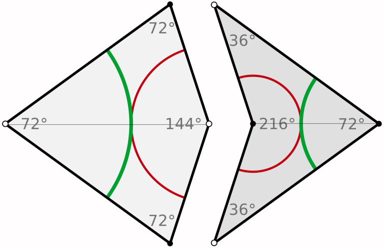
Suppose $\mathcal{P}$ admits a tiling $\mathcal{T}$ which has a translational symmetry through a distance $L$. The composition process may be repeated indefinitely, so we identify successively larger tiles from $\mathcal{T}$ with the translational symmetry. Eventually the incircles of the tiles will have diameter $\geq L$. So it is impossible for a translation through a distance $L$ to be a symmetry of the tiling. $\square$

# Penrose Tiles: The P2 Tiles

- In 1977 Martin Gardner revealed Roger Penrose's "P2" tiling, the Kites and Darts



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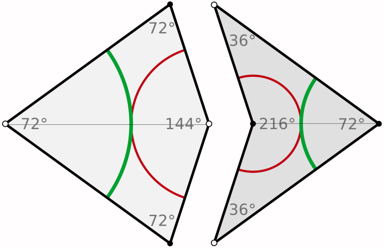


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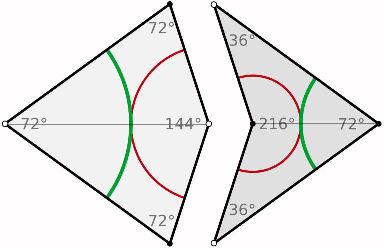


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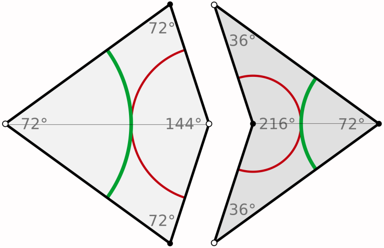


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- The Kites and Darts are an aperiodic set of prototiles.

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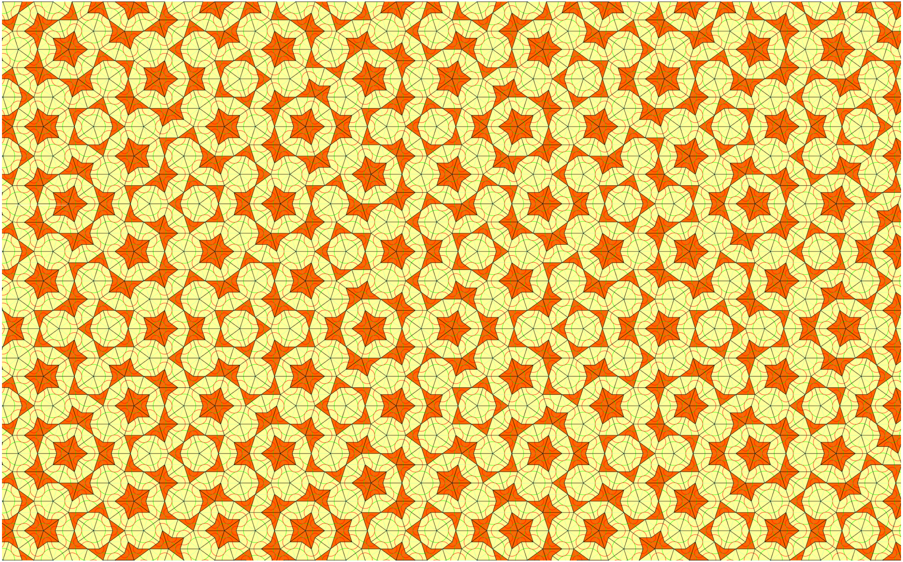

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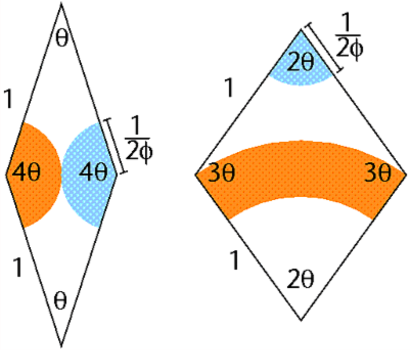

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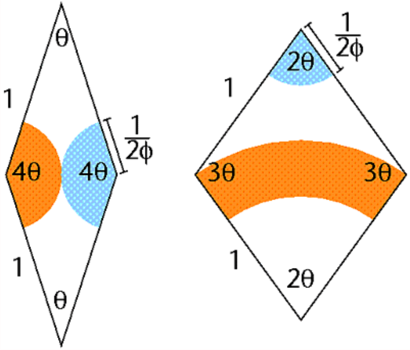
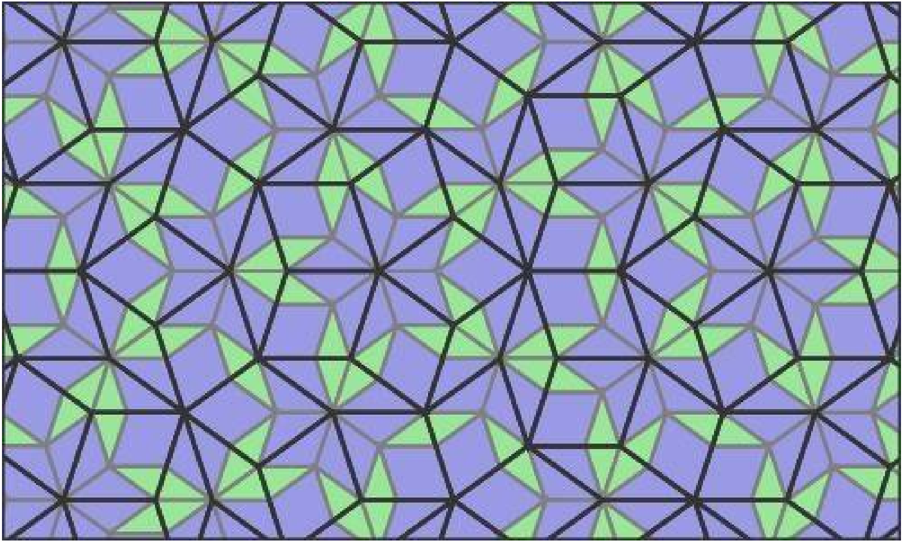



- The P2 and P3 Penrose tiles are *mutually locally derivable*, one can be obtained from the other by a local map.

# Penrose Tiles: A P3 Tiling



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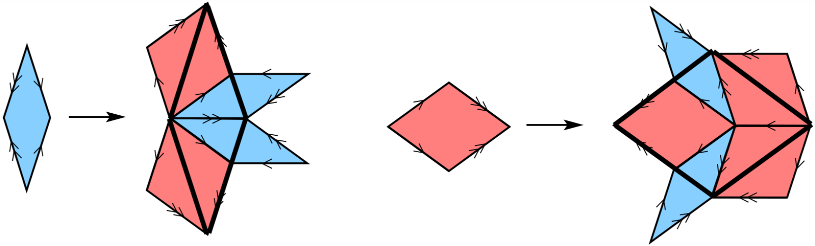

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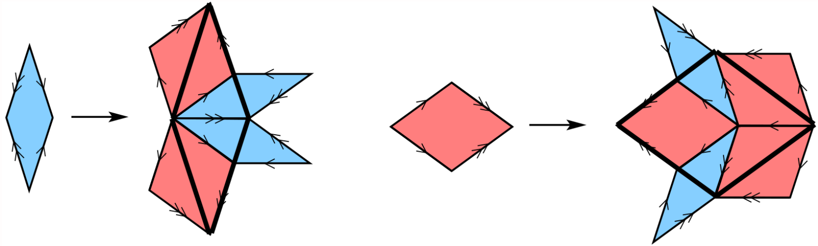

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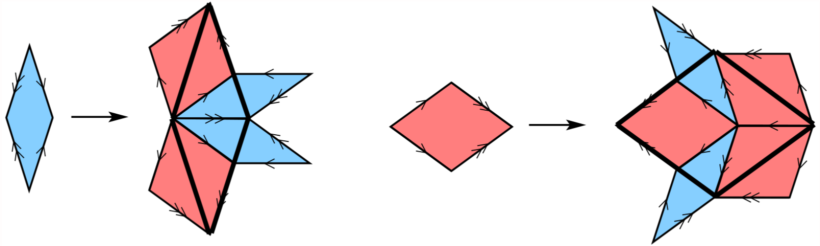
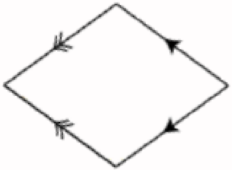


- There is an uncountable number of distinct Penrose tilings
- Each Penrose MLD-class is *locally indistinguishable.* Any finite patch of a Penrose tiling occurs in every other tiling.

# Penrose Tiles: Defected Tilings



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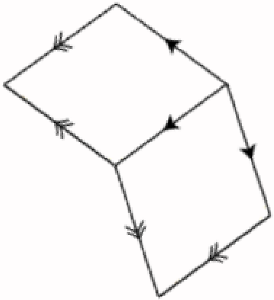

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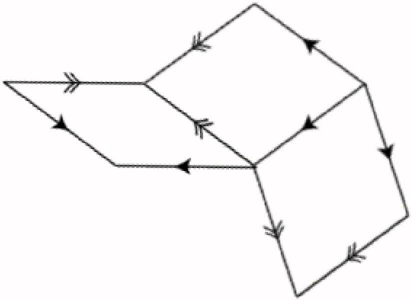

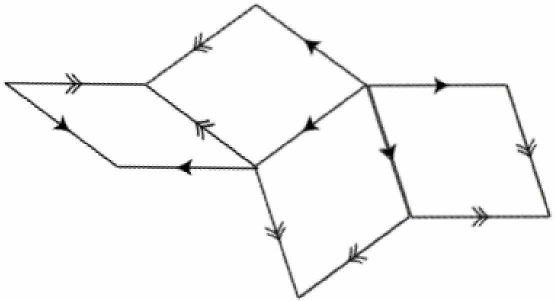

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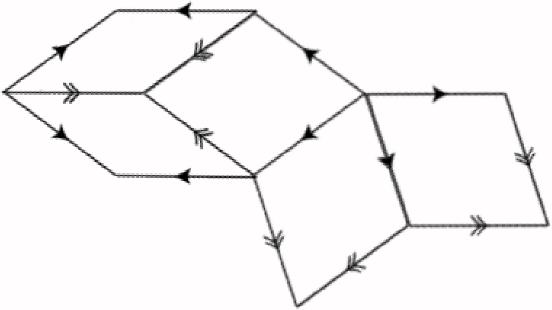

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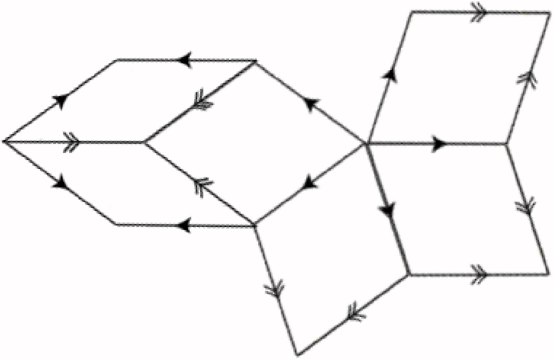

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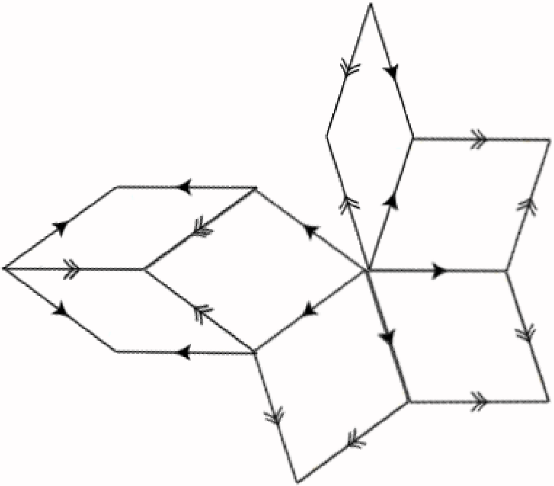

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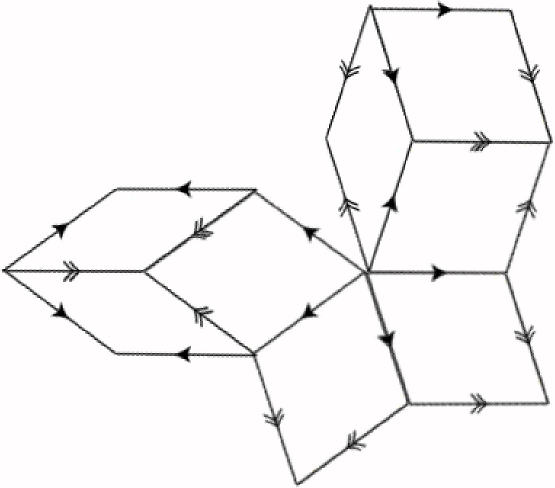

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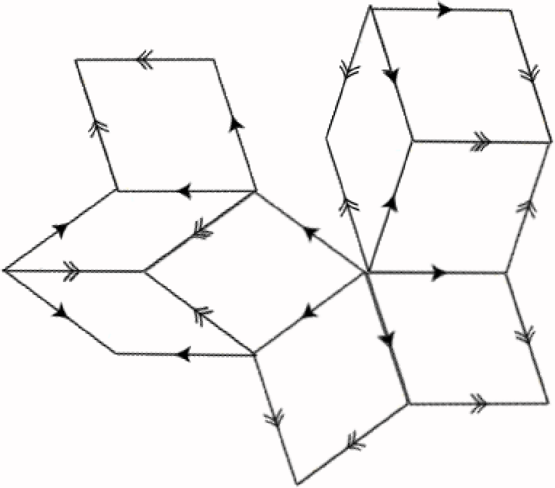

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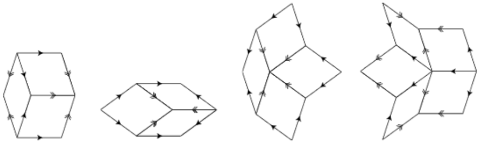
- The only legal configurations around a vertex in a Penrose tiling are



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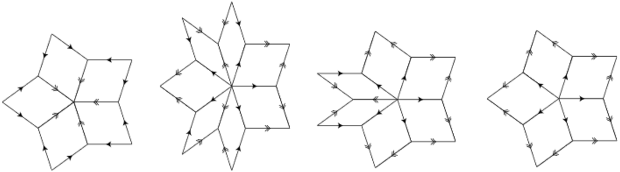

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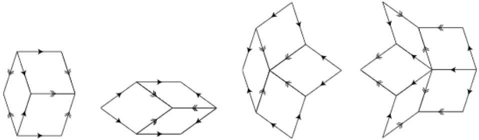

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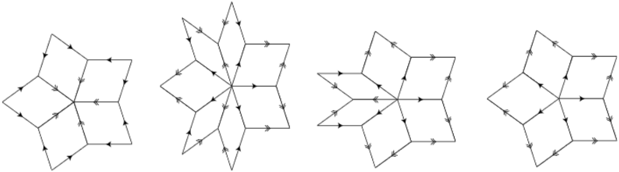

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- A natural quasicrystal cannot adjust itself for the non-locality in laying Penrose tiles

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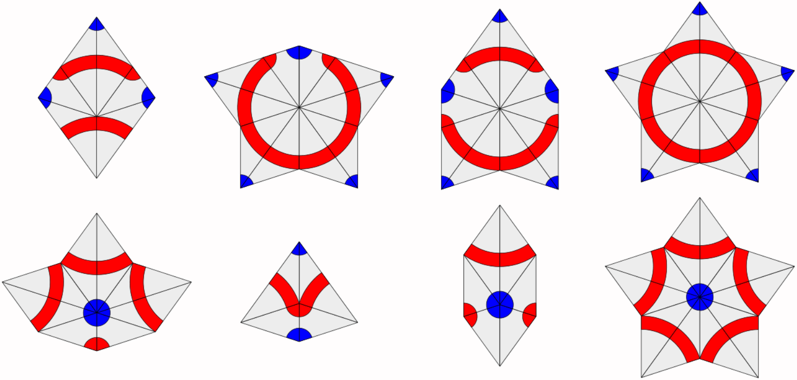
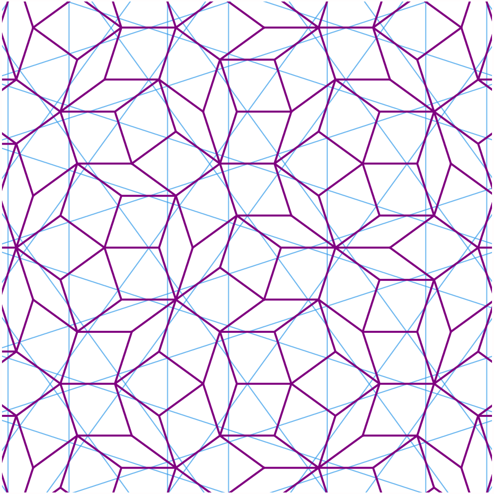


# Ammann Lines: Introduction

- Ammann came up with a marking of Penrose tiles, equivalent to the regular matching rules, now called Ammann lines



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| non-crystallographic root system $\theta^{\parallel}$ | crystallographic partner $\theta$ | degree $N = d/d^{\parallel}$ |
|---|---|---|
| $I_2^p$ ($p$ any prime $\geq 5$) | $A_{p-1}$ | $(p-1)/2$ |
| $I_2^{2^m}$ ($m$ any integer $\geq 3$) | $B_{2^{m-1}}/C_{2^{m-1}}$ | $2^{m-2}$ |
| $I_2^{12}$ | $F_4$ | 2 |
| $I_2^{30}$ | $E_8$ | 4 |
| $H_3$ | $D_6$ | 2 |
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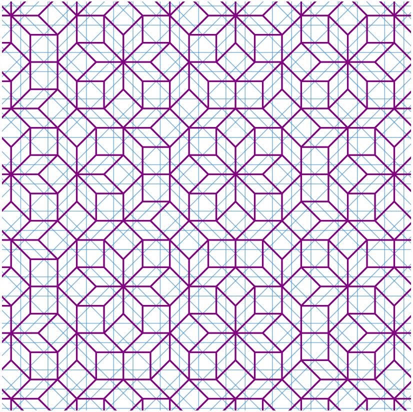
- 8-fold tiling with Ammann lines



# Ammann Lines: Vertex Tiles

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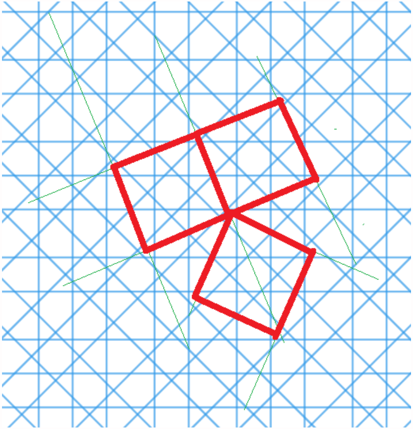

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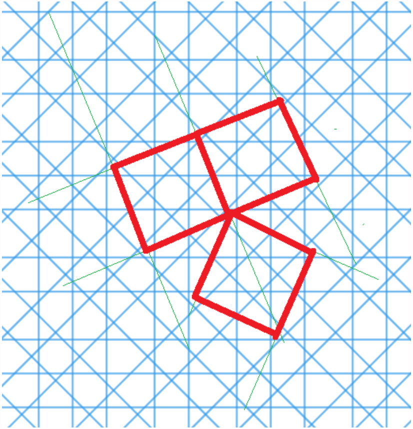

- We introduce a vertex prototile to alleviate these discrepancies. Should it have been there all along?

# Ammann Lines: 8 Fold Tiles

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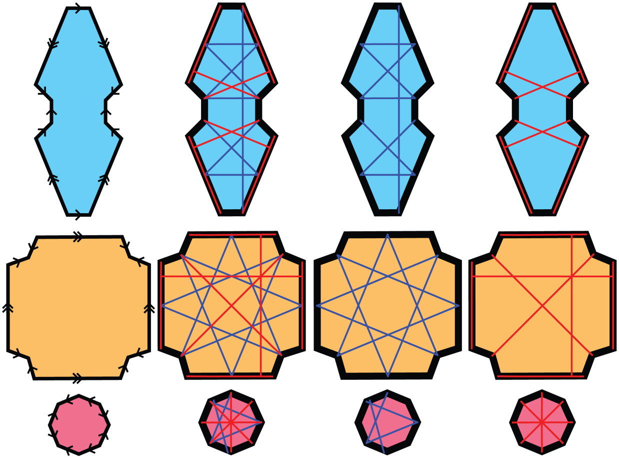

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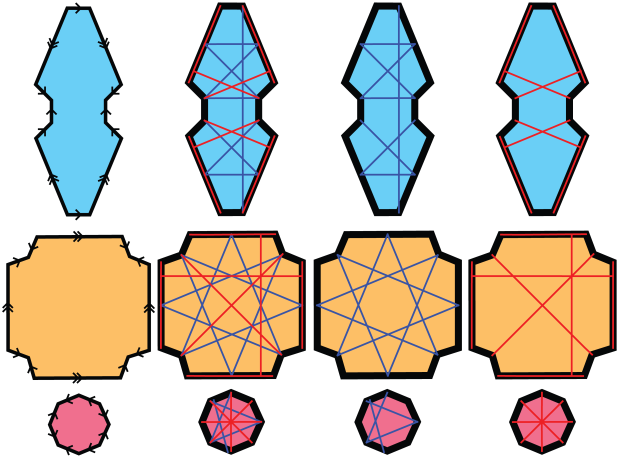
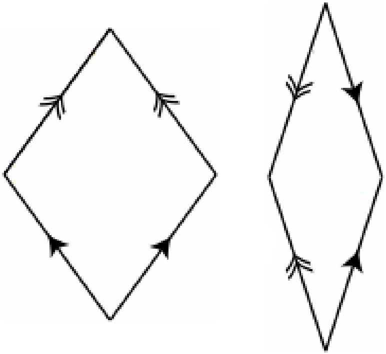


- Are these prototiles equivalent to the regular 8-fold tiles?

# Topology: Introduction

- Treating the matching rule arrows as "charges," the Penrose tiles have no net charge when you travel a path around a tile (and thus a patch).



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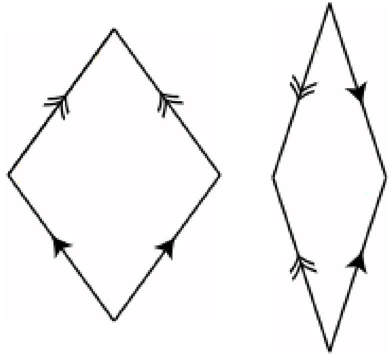
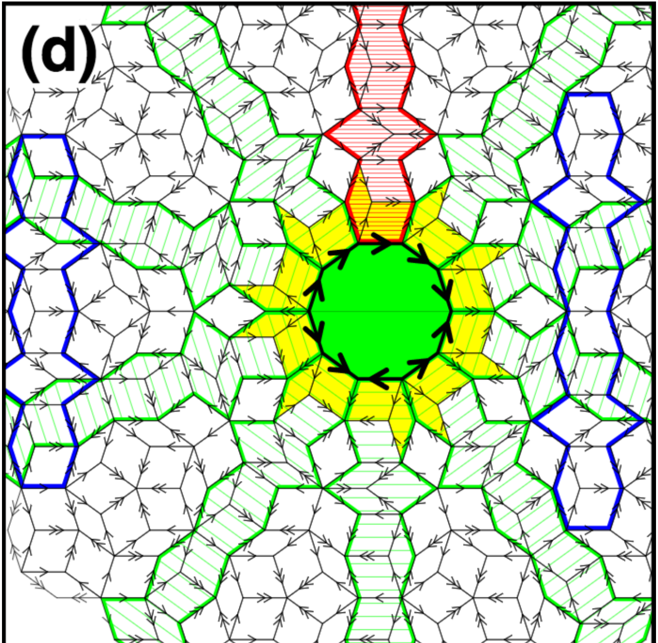


- Topological properties of a defected tiling could lead to interesting math/physics

# Topology: The Decapod



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- Travelling around the decapod we do not accumulate any two-arrow charge, but we accumulate a one-arrow charge of: 10, 8, 6, 4, 2, or 0.
- The decapods cannot be differentiated by their single arrow charge. The decpod count is: 1, 1, 5, 12, 22 and 21 respectively (Pólya necklaces).

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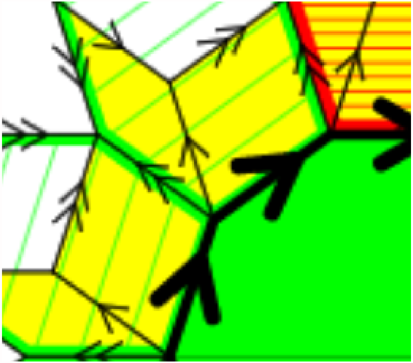
  - Can we construct a vertex tile for the Penrose tiling that adds a new set of charges and lifts the degeneracy on the Decapods?

# Penrose Vertex: Decapod

- Any new decagonal-vertex tile will add new charge to the inner decapod in strings of 4.



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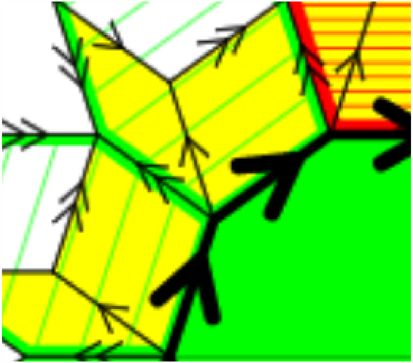

- We work with the theorem (or assumption) that 5 of the vertex-tiles will be face up, and 5 will be face down

# Penrose Vertex: One New Charge

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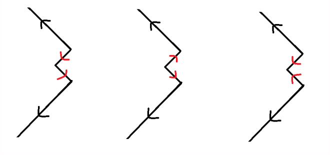

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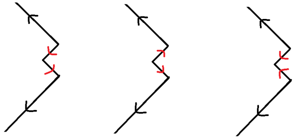

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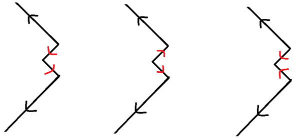

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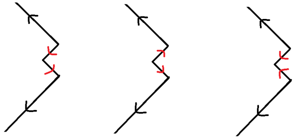

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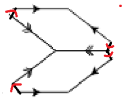
- Also implies that two thin rhombs must be related by a rotation, to get the two red arrows to line up at the right-hand side.



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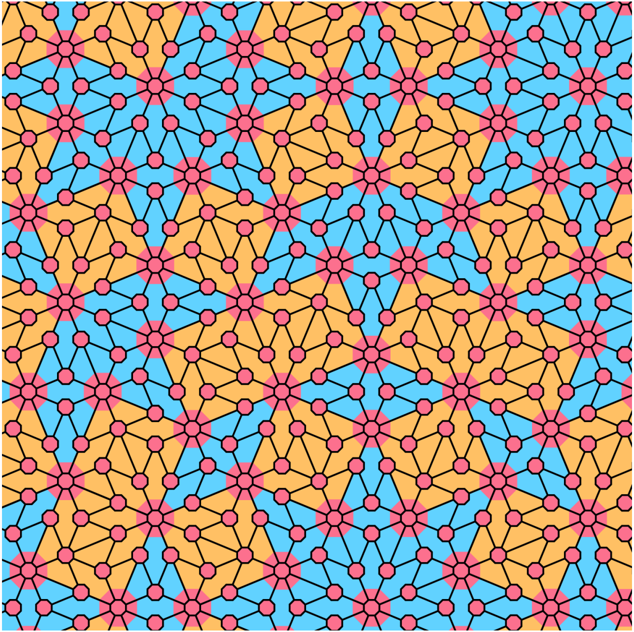
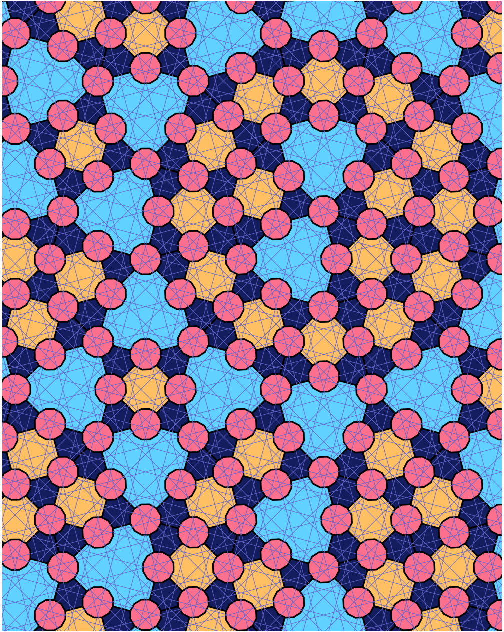
- Are there local matching rules for the 12-fold square-triangle tiling?

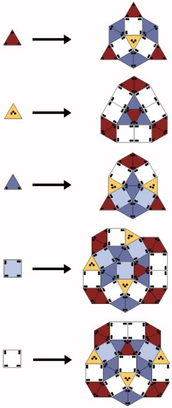
# Ammann 8-Fold Tiling

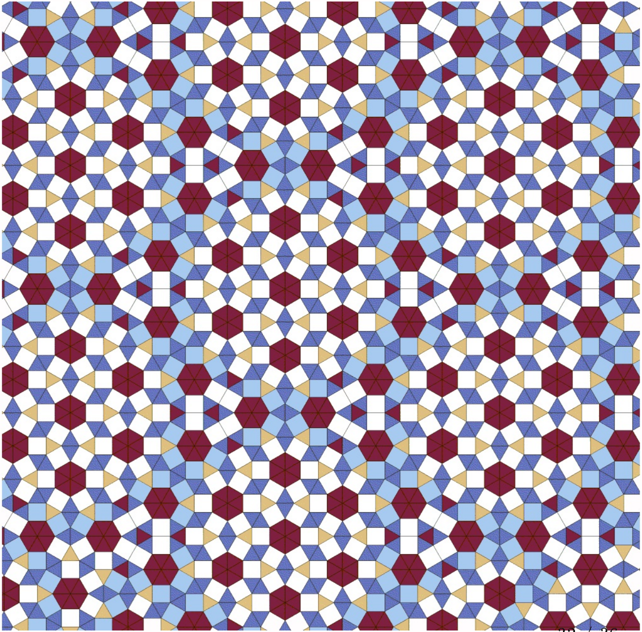


# Ammann 12-Fold Tiling

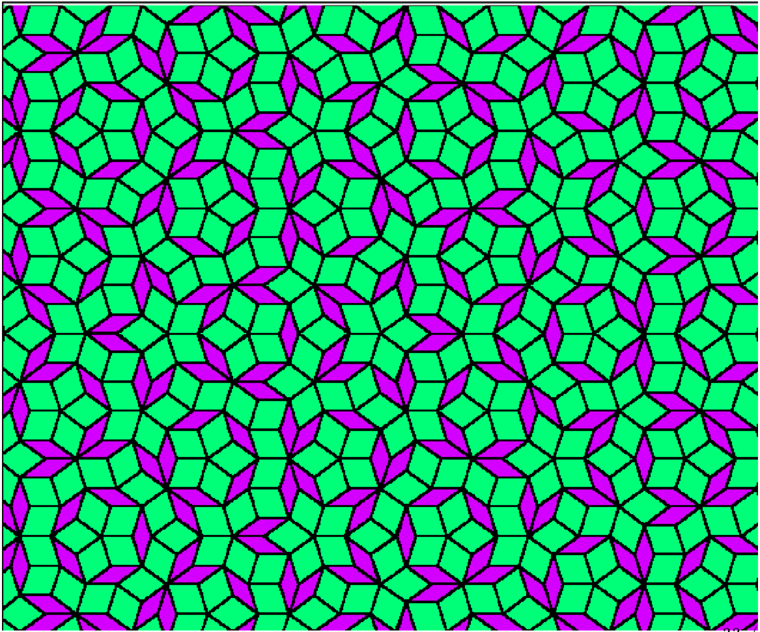


# Square-Triangle Tiling

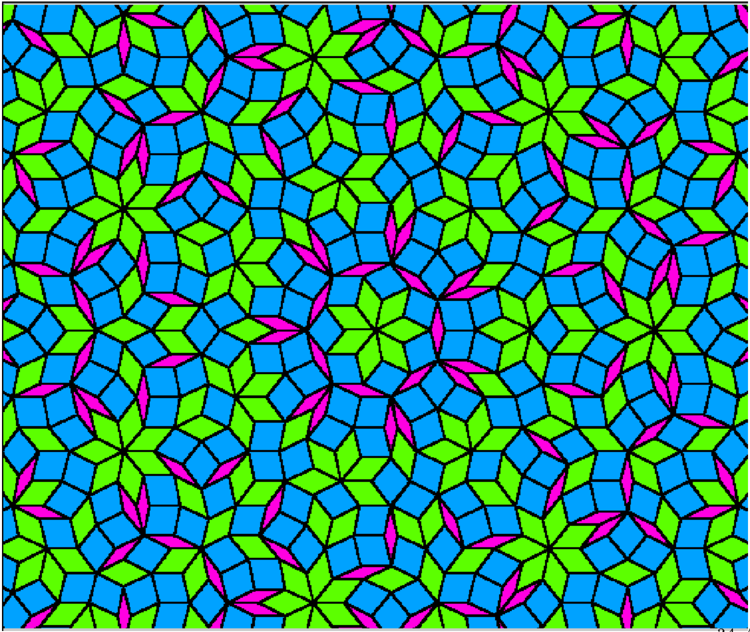




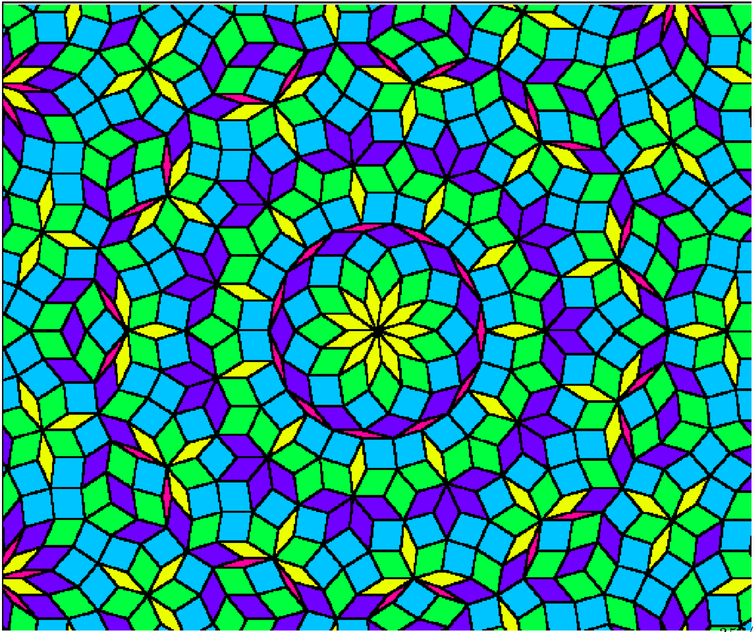
# Cut And Project: 5-Fold (Penrose)



# Cut And Project: 7-Fold



# Cut And Project: 11-Fold



# Cut And Project: 17-Fold