LIMITING FACTORS FOR TREE HEIGHT

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13.8.2018.

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INTRODUCTION

What limits the height to which trees can grow?

What limits the height to which trees can grow?

(a) Hydraulic



(b) Mechanic



INTRODUCTION



Cohesion-tension theory of water transport.

Xylem transports water and solutes from the roots to the leaves.



Mechanics: elastic deflections, self-buckling and beyond Vertical and horizontal stems. Cantilevered and columnar support members.

ALLOMETRY

Assuming that mechanical properties of wood are comparatively uniform:

$$l_{\rm crit} = \left(7.8373 \frac{YI}{\rho gA}\right)^{1/3},$$

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These expressions can be derived without any mechanical assumptions

$$\underbrace{G_T}_{\text{Ial Growth Rate}} = \beta_0 M_L = \beta_1 M_T^{3/4}.$$
 (2)

Annual Growth Rate

 β denotes allometric constants.

More allometric relationships:

$$M_T = M_L + M_S + M_R$$

$$M_L = \beta_2 D^2, \quad M_R = \beta_3 M_S, \quad M_S = \beta_4 D^2 H.$$
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Plugging these relations into eq. (2):

$$H = \beta_5 D^{2/3} - \beta_6, \qquad M_S = \beta_4 \left(\beta_5 D^{8/3} - \beta_6 D^{6/3} \right). \tag{4}$$

ALLOMETRICS IN TREES



 $\beta_5 \& \beta_6$ can not be predicted.

Scalings for herbaceous plants with $\beta_5 = 35.64$ & $\beta_6 = 0.475$. Dashed and solid straight lines are results obtained with the Greenhill-Euler formula.

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 $\beta_5 \& \beta_6$ can not be predicted. Empirical numerical values result in very precise estimates. Greenhill-Euler formula over-estimate values and

presupposes mechanical limitations.

MECHANICAL DAMAGE

TALL TREES: WIND DAMAGE



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So we may expect tall trees to be more susceptible to falling.

This happens in two main ways:

- 1. Uprooting: Shearing at soil-root interface
- 2. Breakage: Excessive bending stress exceeding wood breaking stress

TALL TREES: A MODEL TREE



Sphere of leaves of radius r_L

Mounted to trunk of diameter ${\cal D}$ and height ${\cal H}_m$

Connected to hemisphere of roots

To be safe from uprooting

$$\tau_{\rm wind} = \tau_{\rm root} \tag{5}$$

 $\langle - \rangle$

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m wind}$:

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Working with $\tau_{\rm wind}$:

 $au_{wind} \propto F_{wind}$ $\propto (A_L \, v_{wind}^2 \,) \, H_m$

Aerodynamic drag means wind speed varies as one climbs higher in the atmosphere. In a tree filled region $v_{\rm wind} \propto \sqrt{\rm height}$

$$au_{\rm wind} \propto r_L^2 H_m^2$$

(-)

If M_L is the mass of leaves, then $M_L \propto r_L^3$ It is known in the botany literature that $M_L^2 \propto D^2$ for large plants, giving:

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We use our previously derived result that $H_m \approx H \propto D^{2/3}$ So that

$$\tau_{\rm wind} \propto D^{8/3} \propto M_S \tag{6}$$

For the root system, we have

 $\tau_{\rm root} \propto M_R$

(7)

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$$au_{
m root} \propto M_R$$
 (7)

So that

$$\tau_{\text{wind}} = \tau_{\text{root}}$$

$$\implies M_S \propto M_R \tag{8}$$

Which is a well known result, supporting the proposition that tall trees won't be uprooted.

TALL TREES: A NOTE ON BREAKAGE



Following §19. of L&L's "Theory of Elasticity" and successive exercises we get the EoM

$$\theta'' = -\frac{mg}{EI}\cos\theta \tag{9}$$

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Using empirical facts about breakage under stress, wood properties, it can be shown that critical wind speed for breakage is

$$v_{\rm wind,c} = \sqrt{\frac{\pi}{16} \frac{K_{Ic}}{\delta} \frac{\beta^{3/2}}{\rho_{\rm air} c_d} L^{1/8}} \quad (10)$$
$$\approx 42 {\rm m/s}$$

MODELLING XYLEM

Xylem: plant tissue, transports water and nutrients froom roots to leaves

Approximation: the xylem is a cylindrical tybe of constant radius

We can calculate the rate at which water is transported - **this gives us an estimate for maximum tree height**



We can treat the fluid as continuous

We write down Newton's second law for a small part of the fluid:

$$m\vec{a} = \underbrace{m\vec{f}_{vol.}}_{\text{volume part}} + \underbrace{\Delta\vec{F}_{surf.}}_{\text{surface part}}$$

volume force density:
$$ec{f}_{vol.} = \lim_{\Delta m o 0} rac{\Delta ec{F}}{\Delta m}$$

what about surface forces?



$$\Delta \vec{F}_{surf.} = \oint p \vec{dS}$$

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 $\hat{\Pi}:$ stress tensor, diagonal elements related to pressure

Allows us to go from a surface integral to a volume integral

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Allows us to go from a surface integral to a volume integral

Using Gauss' theorem we get: $\Delta \vec{F}_{surf.} = \oint \hat{\Pi} \vec{n} dS = \int \nabla \hat{\Pi} dV \approx \langle \nabla \hat{\Pi} \rangle \cdot \Delta V$

Finally we get:

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{f}_{vol.} + \frac{\nabla\hat{\Pi}}{\rho}$$

How do we get $\hat{\Pi}$?

$$\hat{\Pi} = -p\hat{I} + \underbrace{\hat{\Lambda}}_{\text{viscosity}}$$

For a large class of fluids we can use a specific form for $\hat{\Lambda}$

When we plug it into the previous equation:

$$ec{a} = rac{dec{v}}{dt} = ec{f}_{vol.} - rac{
abla p}{
ho} + rac{
abla v}{
ho} \Delta ec{v}$$

Our case:

Stationary flow: $\frac{d\vec{v}}{dt} = 0$

Constant pressure gradient along z axis: $\nabla p = K \vec{e_z}$

$$0 = -\frac{K}{\rho} + \frac{\nu}{\rho} \frac{1}{r} \frac{d}{dr} \left(r \frac{dv(r)}{dr} \right)$$

We can solve this equation to get v(r)!

We have:
$$v(r) = \frac{KR^2}{4\nu} \left(1 - \left(\frac{r}{R}\right)^2\right)$$

The rate of flow is: $Q = \int \rho \vec{v} d\vec{S}$

Finally, we get:

$$Q = -\frac{\pi R^4}{8\nu} \cdot \frac{\Delta P}{\Delta h}$$

Any increase in tube length decreases the flow rate! We can approximate Δh_{max} if we know Q_{min} ! Mechanical factors don't limit tree height

But, hydraulic factors do

We can model how water is transported through xylem to arrive at an estimate for maximum tree height

QUESTIONS?

"Maximal plant height and the biophysical factors that limit it", Niklas, 2007 "Tree Physiology

APPENDIX