LIMITING FACTORS FOR TREE HEIGHT

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INTRODUCTION

What limits the height to which trees can grow?

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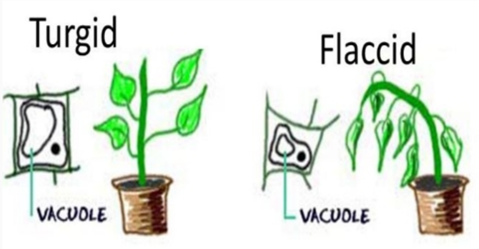
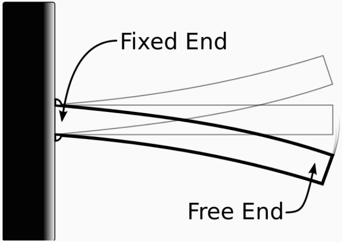
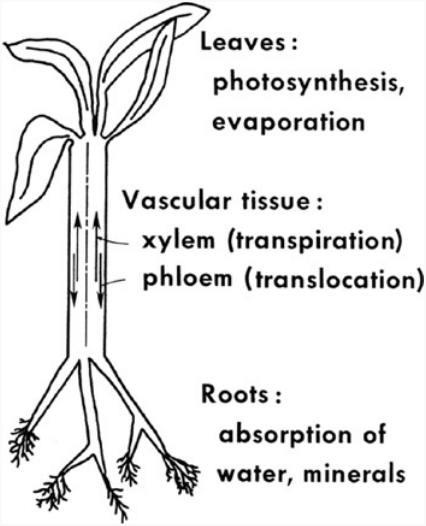
(a) Hydraulic



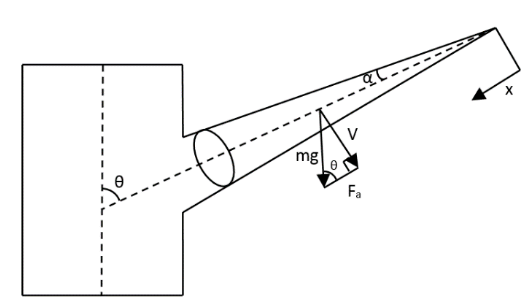
(b) Mechanic





Cohesion-tension theory of water transport.

Xylem transports water and solutes from the roots to the leaves.



Mechanics: elastic deflections, self-buckling and beyond

Vertical and horizontal stems.

Cantilevered and columnar support members.

Allometry

Assuming that mechanical properties of wood are comparatively uniform:

$$\ell_{\text{crit}} = \text{¢:P¨¢¨}\,\left(\frac{3\wedge}{X8}\right)^{\text{o=¨}}; \qquad (1)$$

$$[\;/\; K^{\text{©=¨}}; \quad g \;/\; K^{\text{P=¨}}:$$

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$$[\ / \ K^{©=¨}; \quad g \ / \ K^{P=¨}:$$

These expressions can be derived without any mechanical assumptions

$$\underbrace{W}_{\text{Annual Growth Rate}} = \cdot\, g_{d} = {}_{o}\, g^{¨=V}: \qquad (2)$$

denotes allometric constants.

More allometric relationships:

$$g_{¦} = g_{d} + g_{\sim} + g_{|}$$
$$g_{d} = {}_{©}K^{©}; \quad g_{|} = {}^{¨}g_{\sim}; \quad g_{\sim} = {}_{V}K^{©}[: \tag{3}$$

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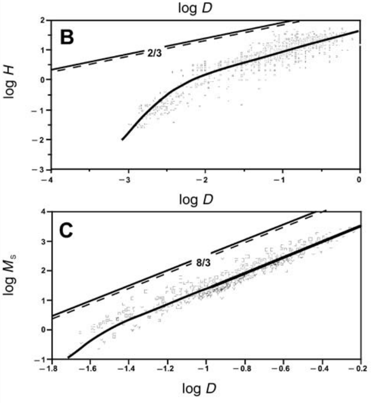
Plugging these relations into eq. (2):

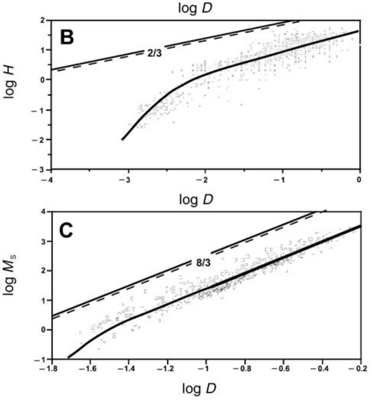
$$[= {}_{U}K^{©=¨} \quad {}_{£}; \qquad g_{\sim} = {}_{V} \quad {}_{U}K^{P=¨} \quad {}_{£}K^{£=¨} \quad : \tag{4}$$



Scalings for herbaceous plants with U = ¨U:£V & £ = ·:V¢U. Dashed and solid straight lines are results obtained with the Greenhill-Euler formula.

U & £ can not be predicted.

ALLOMETRICS IN TREES



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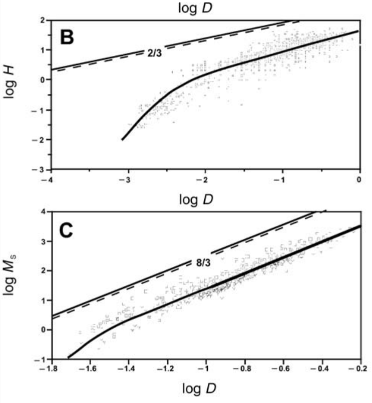
Empirical numerical values result in very precise estimates.



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Empirical numerical values result in very precise estimates.

Greenhill-Euler formula over-estimate values and presupposes mechanical limitations.

Mechanical Damage

TALL TREES: WIND DAMAGE

Is it dangerous to be a tall tree?

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Trees fail mechanically only due to wind.

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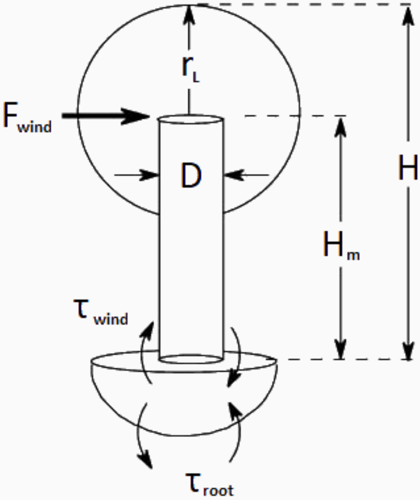
This happens in two main ways:

1. Uprooting: Shearing at soil-root interface
2. Breakage: Excessive bending stress exceeding wood breaking stress



Sphere of leaves of radius $\}_d$

Mounted to trunk of diameter K and height $[_h$

Connected to hemisphere of roots

To be safe from uprooting

$$°_{j}L = \}nn§ \tag{5}$$

TALL TREES: UPROOTING

To be safe from uprooting

$$∘_{jL} = \}nn§ \tag{5}$$

Working with ∘_jL:

$$∘_{jL} \;/\; S∘_{jL}$$
$$/\; (8_d ®^{©}∘_{jL})\; [\; h$$

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$$/\;(8_d ®^{©}∘_{jL})\;[_h$$

Aerodynamic drag means wind speed varies as one climbs higher in the atmosphere. In a tree filled region $®∘_{jL} \;/\; p\overline{\backslash 0\_X\backslash §}$

$$∘_{jL} \;/\; \}^{©}_{d}\;[^{©}_{h}$$

If g_d is the mass of leaves, then $g_d \;/\; \}_d^{¨}$

It is known in the botany literature that $g_d^{©} \;/\; K^{©}$ for large plants, giving:

$$\}_d^{©} \;/\; K^{V=¨}$$

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We use our previously derived result that $[_h \quad [\ / \ K^{©=¨}$

So that

$$\circ\_jL \ / \ K^{P=¨} \ / \ g_{\sim} \tag{6}$$

For the root system, we have

$$\}nn\S \;/\; g \;| \tag{7}$$

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$$\}nn\S \;/\; g_{\,|} \tag{7}$$

So that

$$\circ\_jL = \}nn\S$$
$$\Longrightarrow \quad g_{\sim} \;/\; g_{\,|} \tag{8}$$

Which is a well known result, supporting the proposition that tall trees won’t be uprooted.

Following §19. of L&L's "Theory of Elasticity" and successive exercises we get the EoM

$$ {}^{00} = \frac{h\,X}{N^{\wedge}} \cos \tag{9} $$

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Using empirical facts about breakage under stress, wood properties, it can be shown that critical wind speed for breakage is

$$ ® \circ_{\_j \, LJH} = s \overline{\frac{}{o£} \frac{b_{\wedge H}}{} \frac{{}^{\cdot\cdot =©}}{9_{\_\}} H_L} d^{o=P}} \tag{10} $$

V© h ¤

Modelling xylem

Xylem: plant tissue, transports water and nutrients froom roots to leaves

Approximation: the xylem is a cylindrical tybe of constant radius

We can calculate the rate at which water is transported - **this gives us an estimate for maximum tree height**

We can treat the fluid as continuous

We write down Newton's second law for a small part of the fluid:

$$h9 = \underbrace{h\mathcal{T}_{\text{®}ne:}}_{\text{volume part}} + \underbrace{\mathfrak{S}_{\text{«}\}T:}}_{\text{surface part}}$$

volume force density: $\mathcal{T}_{\text{®}ne:} = \lim_{\Delta h\,!\,\cdot} \frac{\mathfrak{S}}{h}$

what about surface forces?

STRESS TENSOR

$$\mathfrak{S}_{\text{«}\}T:} = qL\tilde{}\sim$$

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$$\mathfrak{S}_{«\}T:} = qL\sim = \underbrace{\wedge\,|\{z\}}_{\text{stress tensor}} \; jL\sim$$

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$$\mathfrak{S}_{\ll \}T:} = q\tilde{L}\sim = \underbrace{\overset{\wedge}{\{z\}}}_{\text{stress tensor}} \; \dot{\jmath} L\sim$$

$\wedge$: stress tensor, diagonal elements related to pressure

Allows us to go from a surface integral to a volume integral

$$\mathfrak{S}_{«\}T:} = qL\sim = \underbrace{\wedge}_{\text{stress tensor}} \jmath L\sim$$

$\wedge$: stress tensor, diagonal elements related to pressure

Allows us to go from a surface integral to a volume integral

Using Gauss' theorem we get:

$$\mathfrak{S}_{«\}T:} = \wedge \jmath L\sim = r \wedge L - \quad h\, r \wedge i \quad -$$

Finally we get:

$$9 = \frac{L®}{L§} = \mathcal{T}_{®ne:} + \frac{r \wedge}{}$$

How do we get ^ ?

$$\wedge = q\wedge + \underbrace{\overset{\wedge}{|\{z\}}}_{\text{viscosity}}$$

For a large class of fluids we can use a specific form for ^

When we plug it into the previous equation:

$$9 = \frac{L®}{L§} = T®ne: \frac{r\, q}{} + \frac{\overbrace{z\}|\{}^{\text{viscosity coeff.}}}{} ®$$

Our case:

Stationary flow: $\frac{L\text{®}}{L\text{§}} = \cdot$

Constant pressure gradient along ¶ axis: $r\,q = b\,\theta_{\text{¶}}$

$$\cdot = \frac{b}{} + -\frac{\text{o}}{\}}\frac{L}{L\}} \quad \}\frac{L\text{®}(\})}{L\}}$$

We can solve this equation to get ®(})!

MAXIMUM HEIGHT APPROXIMATION

We have: ®(}) $=$ $\frac{b\,|^{2}}{V}$ o ($\frac{\}}{|}$)©

The rate of flow is: w $=$ ®L~~

Finally, we get:

$$w = \frac{|^{V}}{P} \; \frac{p}{\backslash}$$

Any increase in tube length decreases the flow rate!

We can approximate \ $_{h\,9^{2}}$ if we know w_{h_j}!

CONCLUSION

Mechanical factors don’t limit tree height

But, hydraulic factors do

We can model how water is transported through xylem to arrive at an estimate for maximum tree height

Questions?

”Maximal plant height and the biophysical factors that limit it”, Niklas, 2007 ”Tree Physiology

APPENDIX