Quasicrystals in Physics

Justin Kulp

January, 21st, 2019

Background: Tiling

Definition (Tiling)

A *tiling* of $\mathbb{R}^d$ is a non-empty countable collection of closed sets in $\mathbb{R}^d$, $\mathcal{T} = \{T_i : i \in I\}$, subject to the constraints that:

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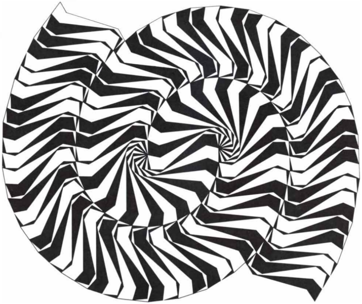
- The *symmetries* of $\mathcal{T}$ are isometries that map $\mathcal{T}$ onto itself, and $\mathcal{T}$ is *nonperiodic* if it has no translational symmetry.

Background: Tiling

Periodic tiling by M.C. Escher

Nonperiodic tiling by Heinz Voderberg



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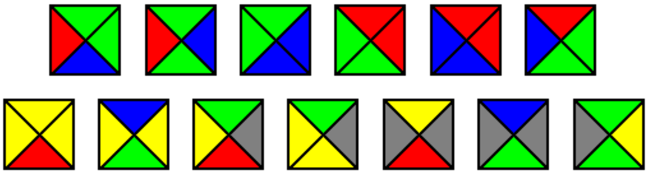

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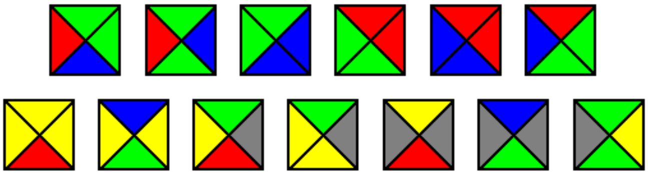

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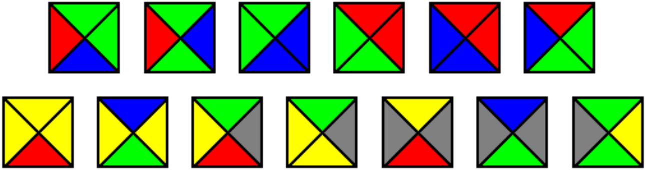

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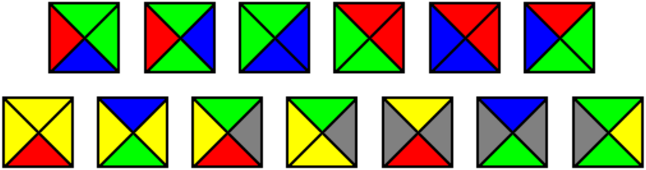

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- A set of prototiles that only admits nonperiodic tilings is called *aperiodic*.

Background: Quasicrystals

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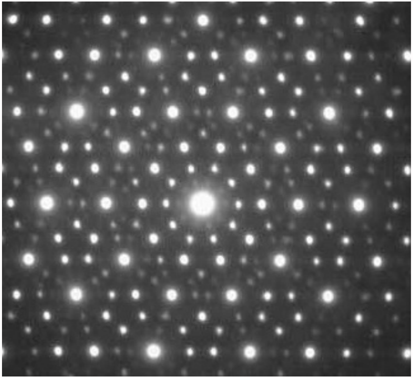

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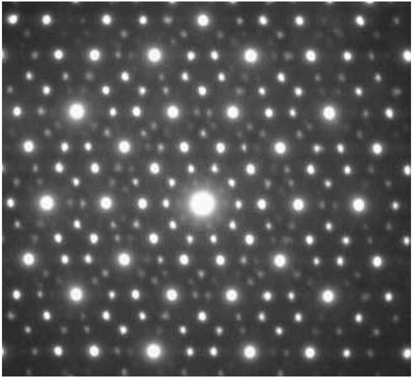

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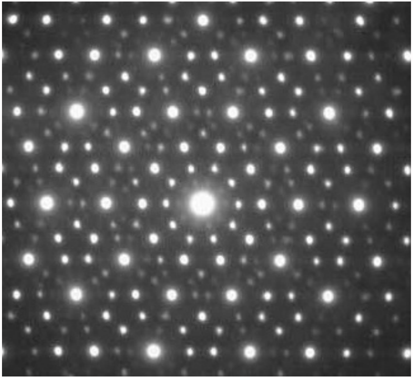
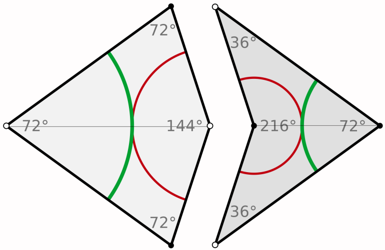


  - Classically forbidden diffraction pattern.
  - Explainable as the diffraction of a lattice described by a quasiperiodic function: $\sin(x) + \sin(\tau x)$.

Penrose Tiles: The P2 Tiles

- In 1977 Martin Gardner revealed Roger Penrose's "P2" tiling, the Kites and Darts



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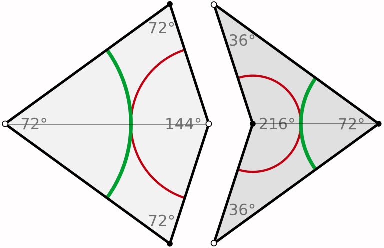

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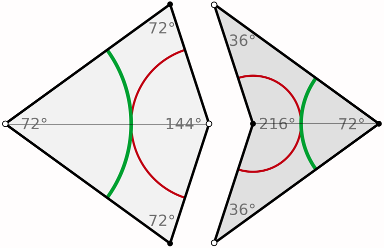

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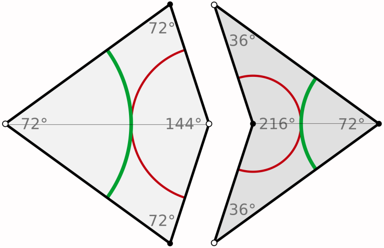
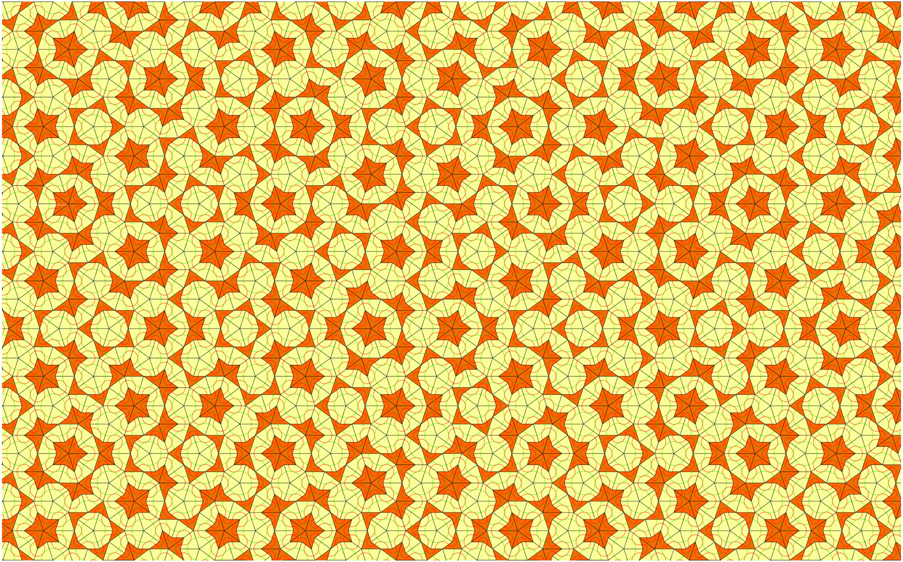
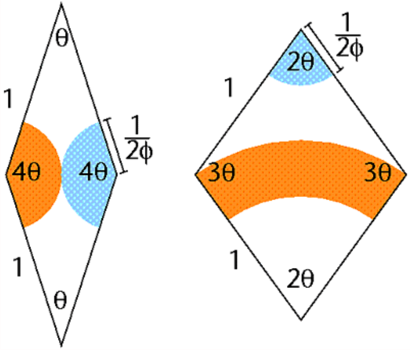


  - The tiles are free to rotate/flip.
  - Subject to matching rule that black and white vertices join, or red and green lines go unbroken (Robinson's rules).
- The Kites and Darts are an aperiodic set of prototiles.

Penrose Tiles: A P2 Tiling



- The Penrose P2 tiles are equivalent to the P3 Penrose tiles



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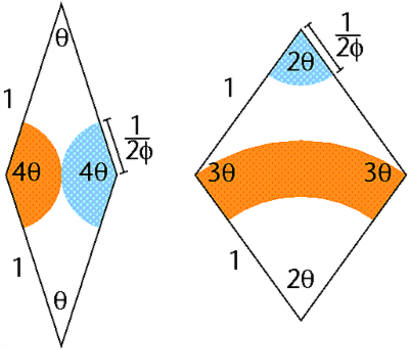
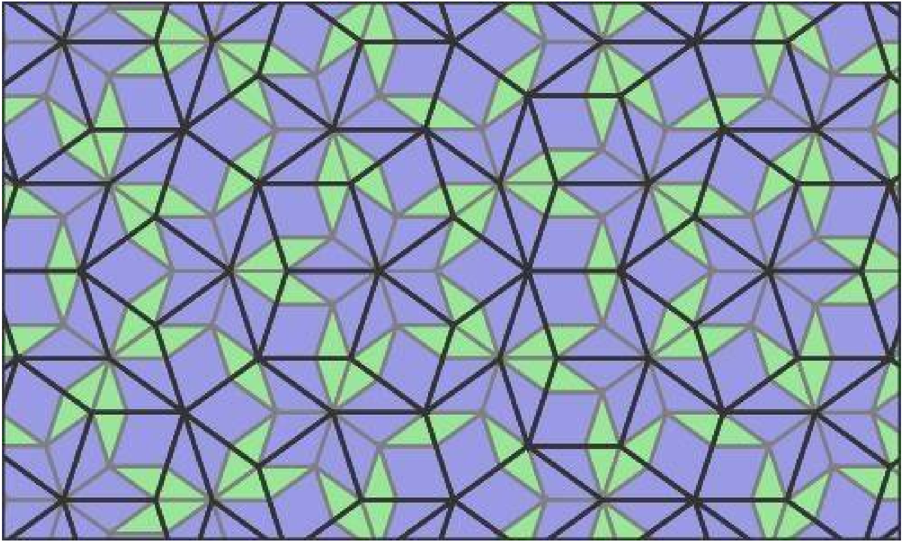


- The P2 and P3 Penrose tiles are *mutually locally derivable*, one can be obtained from the other by a local map.

Penrose Tiles: A P3 Tiling



Penrose Tiles: Properties

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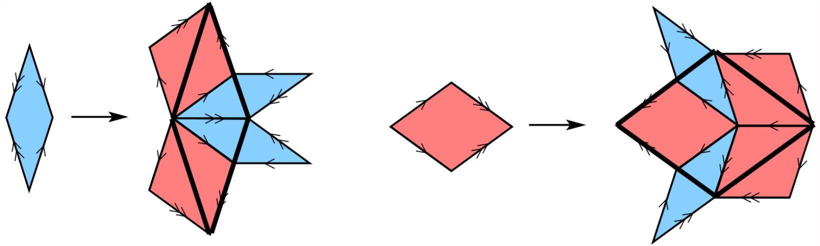

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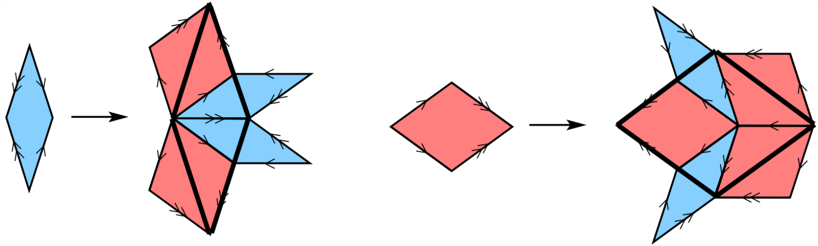

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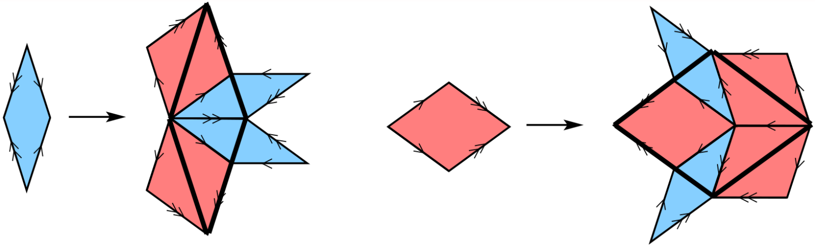

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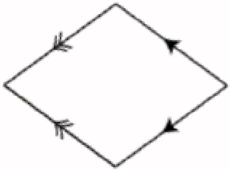
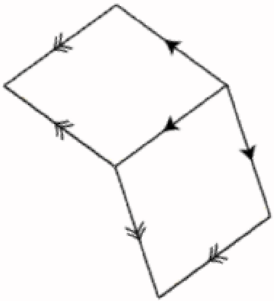
- Each Penrose MLD-class is *locally indistinguishable*. Any finite patch of a Penrose tiling occurs in every other tiling.



Penrose Tiles: Defected Tilings



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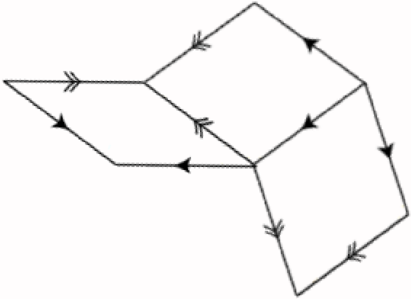

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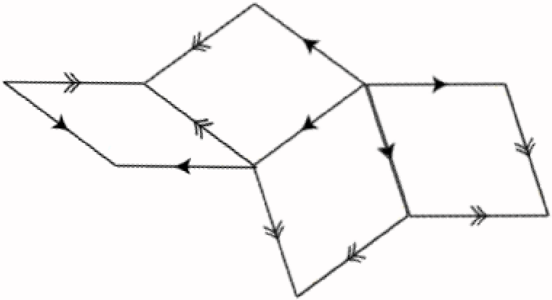

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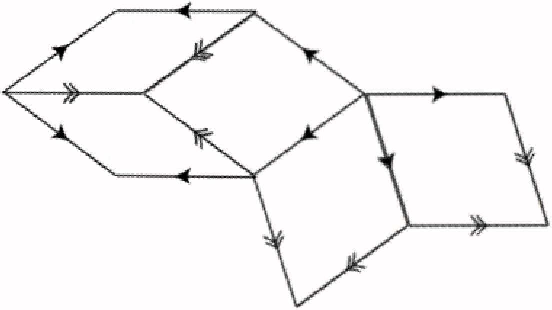

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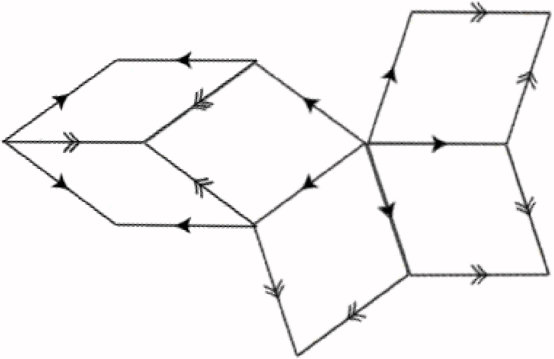

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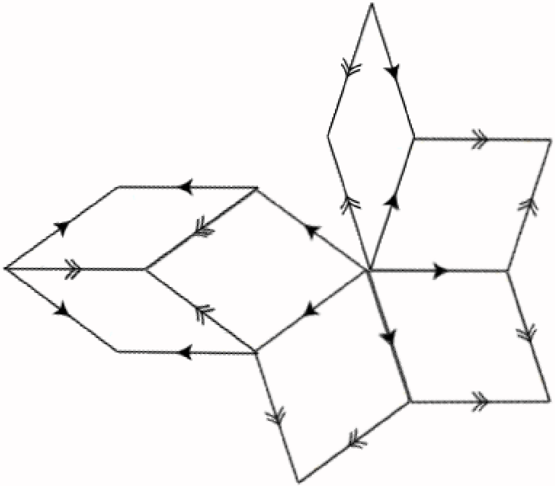

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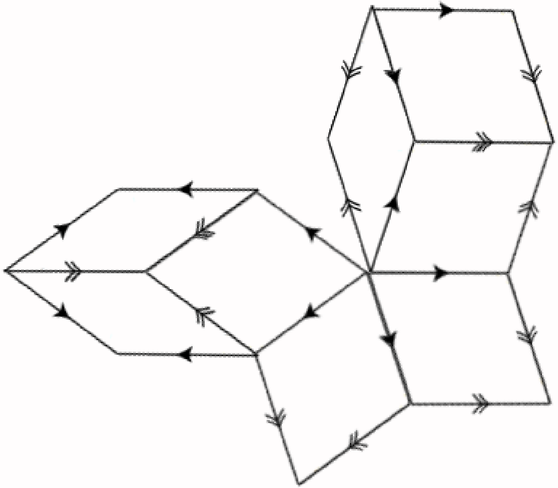
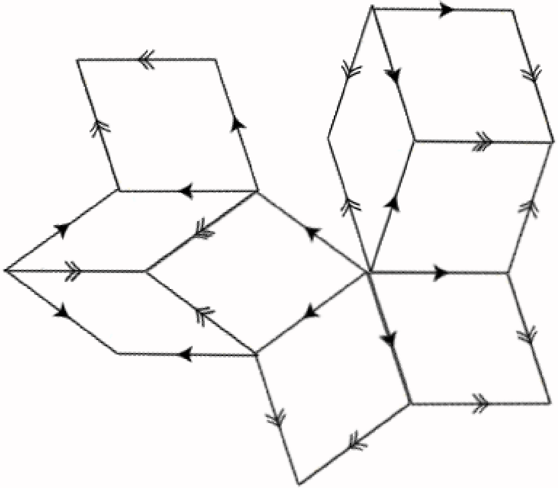
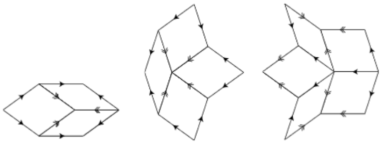


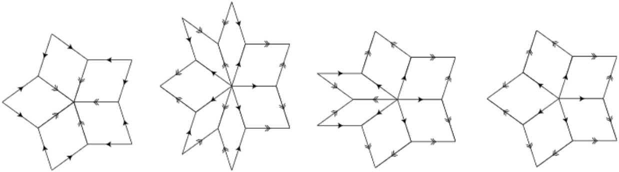


Penrose Tiles: Legal Vertices

- The only legal configurations around a vertex in a Penrose tiling are



(a) The D. (b) The Q. (c) The K. (d) The J.



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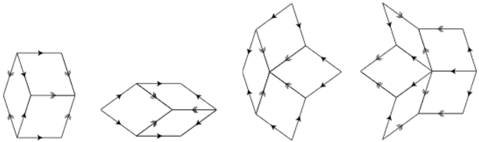

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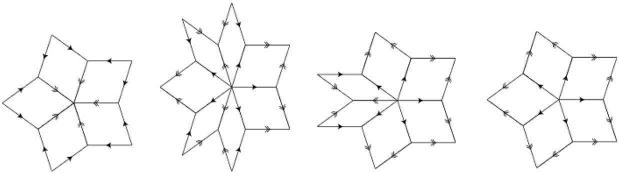

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- A natural quasicrystal cannot adjust itself for the non-locality in laying Penrose tiles

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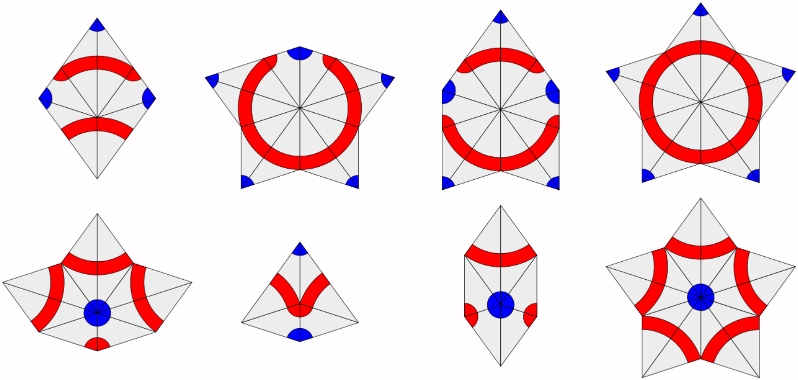
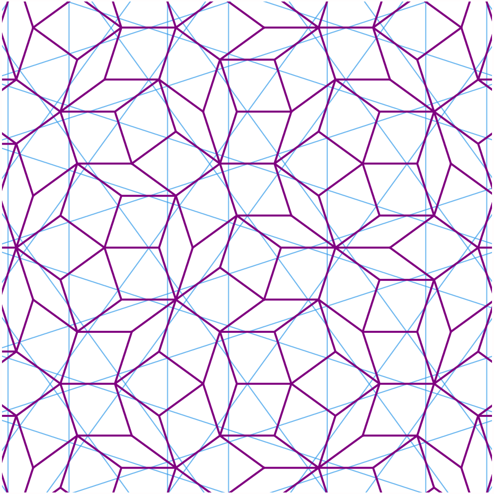


Ammann Lines: Introduction

- Ammann came up with a marking of Penrose tiles, equivalent to the regular matching rules, now called Ammann lines



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non-crystallographic root system $\theta^{\parallel}$	crystallographic partner θ	degree $N = d/d^{\parallel}$
I_2^p (p any prime ≥ 5)	A_{p-1}	$(p-1)/2$
$I_2^{2^m}$ (m any integer ≥ 3)	$B_{2^{m-1}}/C_{2^{m-1}}$	2^{m-2}
I_2^{12}	F_4	2
I_2^{30}	E_8	4
H_3	D_6	2
H_4	E_8	2

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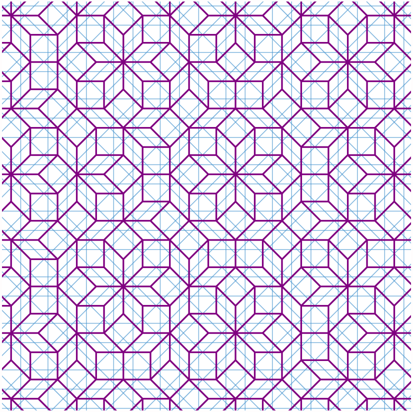
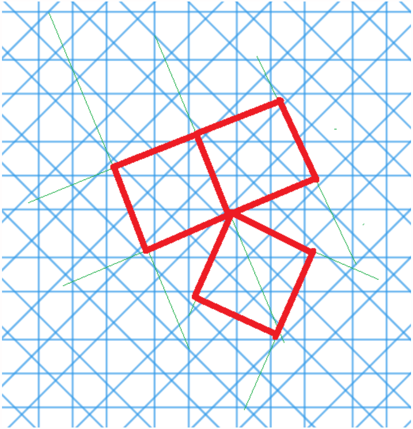
- 8-fold tiling with Ammann lines



Ammann Lines: Vertex Tiles

- When we reconfigure the tiles on the Ammann lines in a different way, vertices mark the tiles in very different ways.



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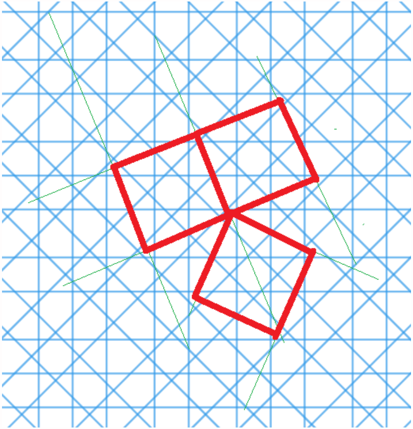
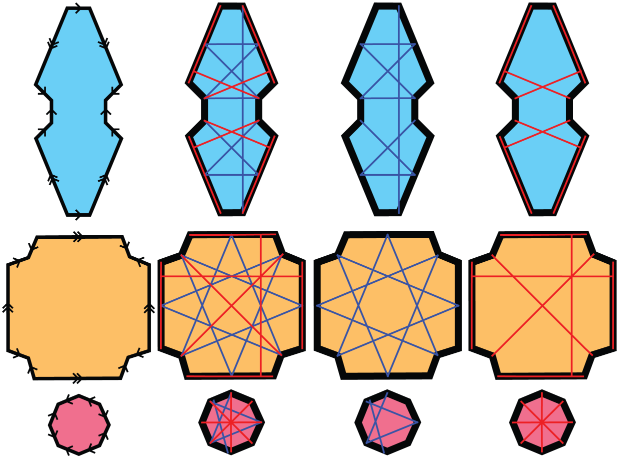


- We introduce a vertex prototile to alleviate these discrepancies. Should it have been there all along?

Ammann Lines: 8 Fold Tiles

- The 8-fold Ammann lines effectively force us to remark our square/rhomb/vertex tile as follows



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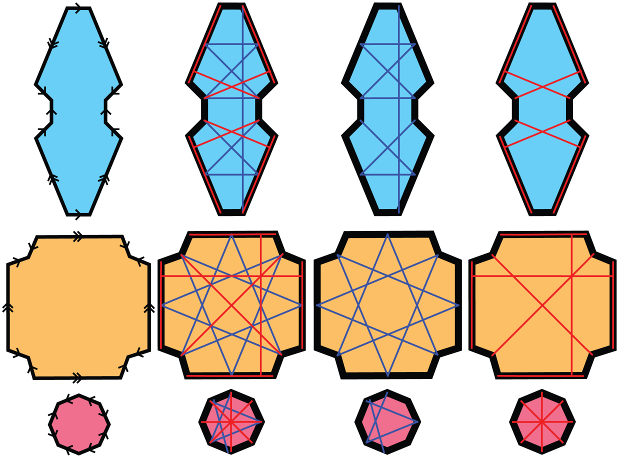
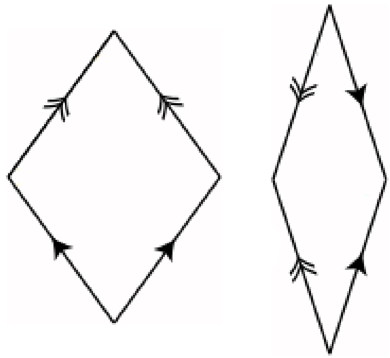


- Are these prototiles equivalent to the regular 8-fold tiles?

Topology: Introduction

- Treating the matching rule arrows as “charges,” the Penrose tiles have no net charge when you travel a path around a tile (and thus a patch).



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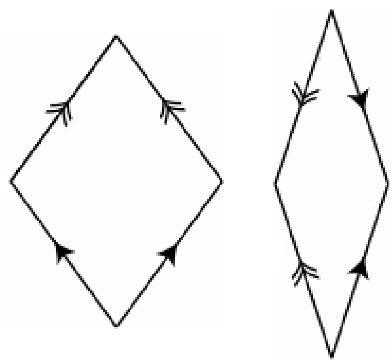
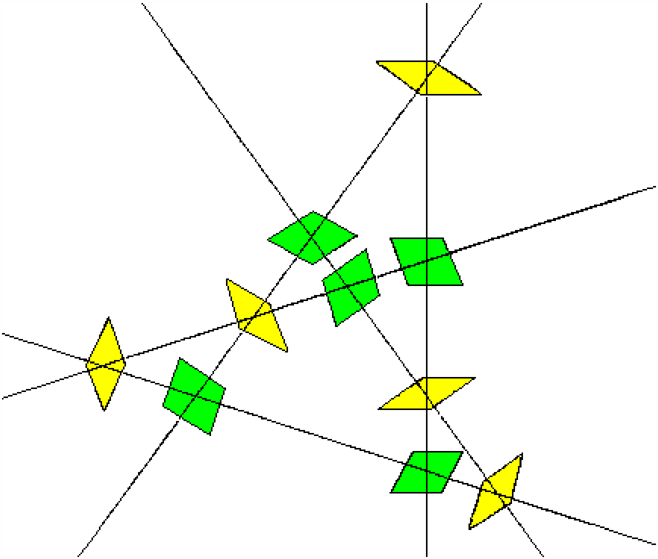


- Topological properties of a defected tiling could lead to interesting math/physics

de Bruijn Multigrids



Degenerate Structure

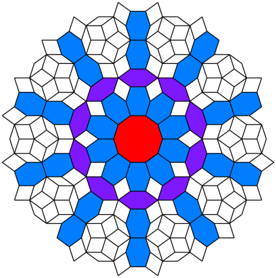


Figure 4.2: The orthogonal dual of the singular pentagrid with $\gamma_j = 0$ for all j.

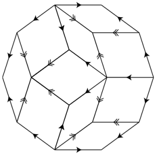
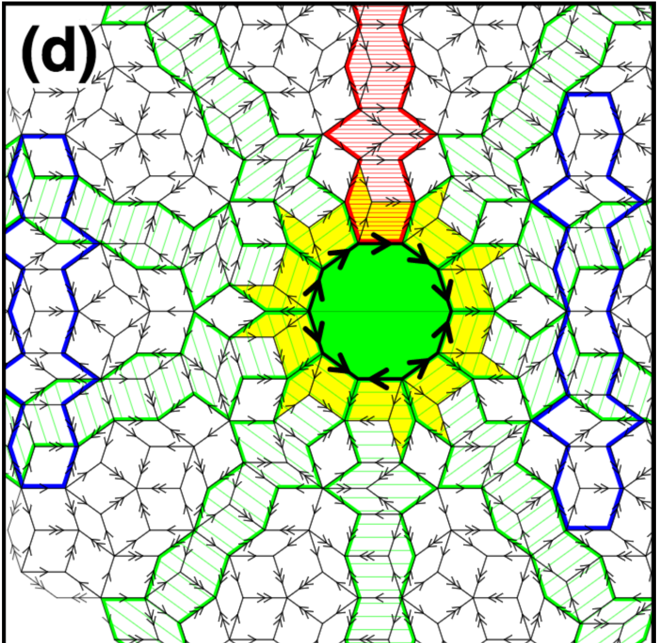


Figure 4.3: The decagon in Figure 4.2 corresponds to the ten possible rotations of this patch.

Topology: The Decapod



Topology: The Decapod

- Defected tiling, pointed out by John Conway. Can be seen as one of the most defected tilings (via Ammann lines or Pentagrid construction).

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- The decapods cannot be differentiated by their single arrow charge. The decpod count is: 1, 1, 5, 12, 22 and 21 respectively (Pólya necklaces).

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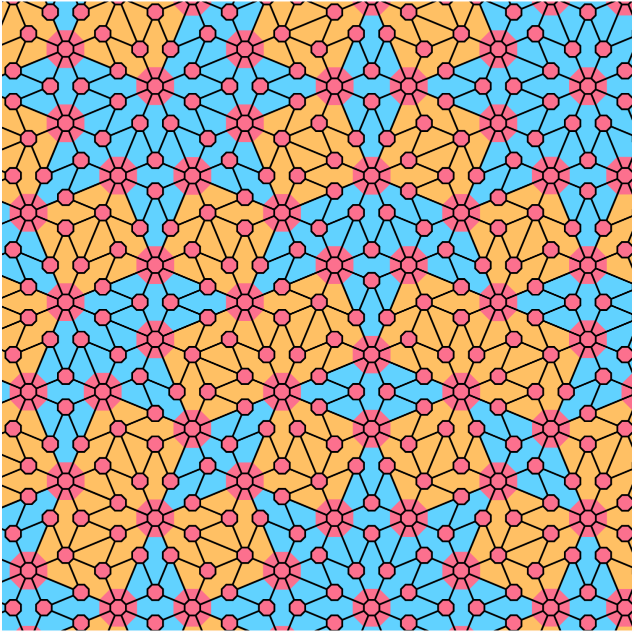
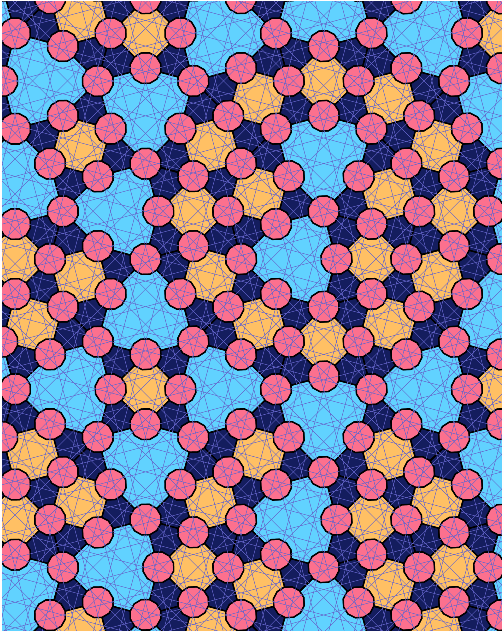
- Are there local matching rules for the 12-fold square-triangle tiling?

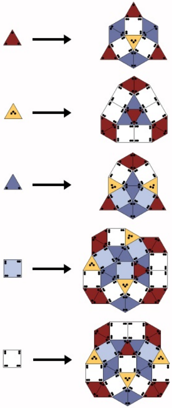
Ammann 8-Fold Tiling

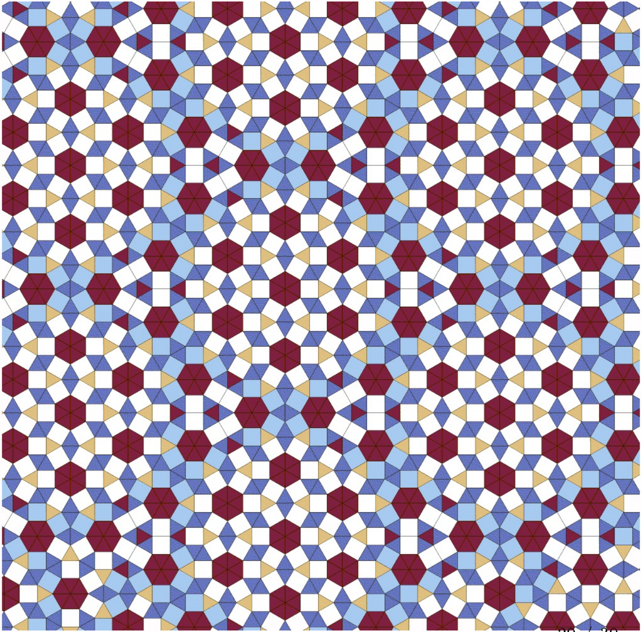


Ammann 12-Fold Tiling

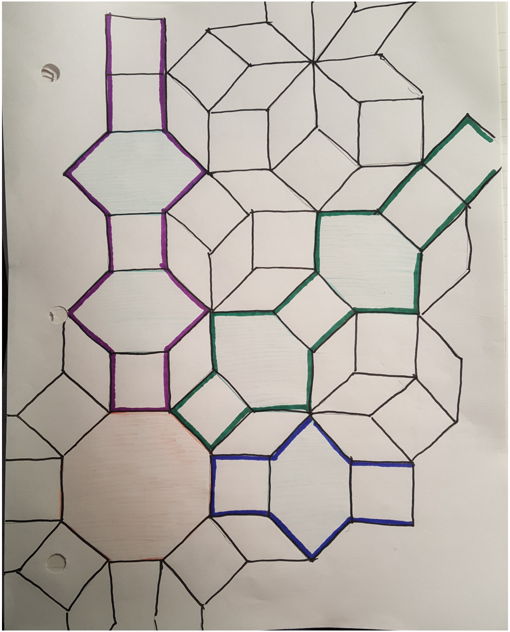


Square-Triangle Tiling

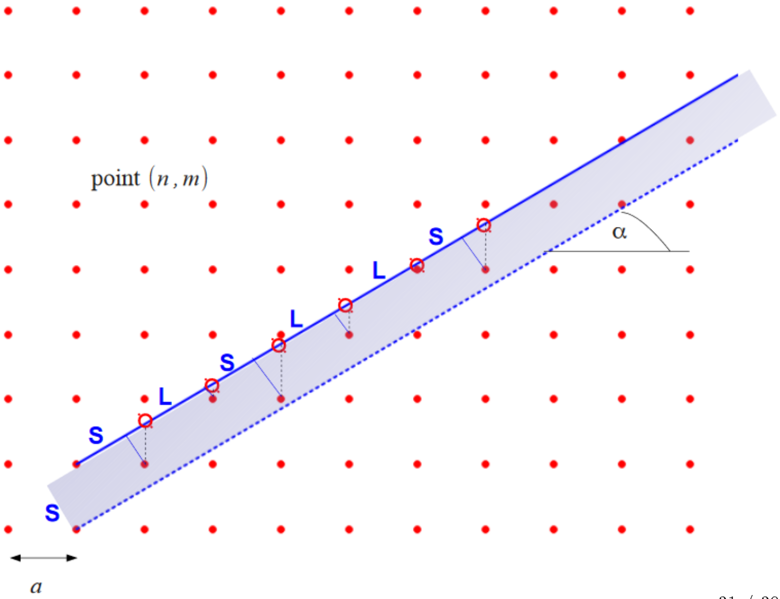




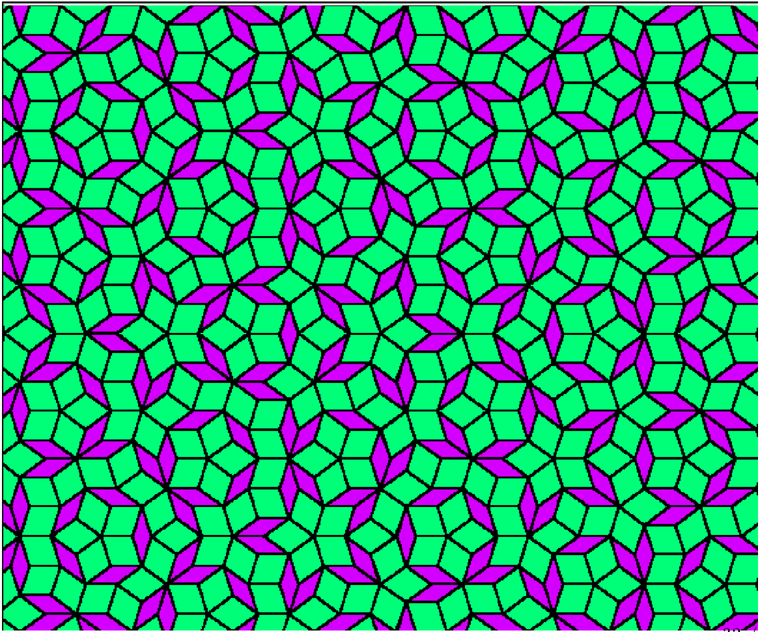
Octopod



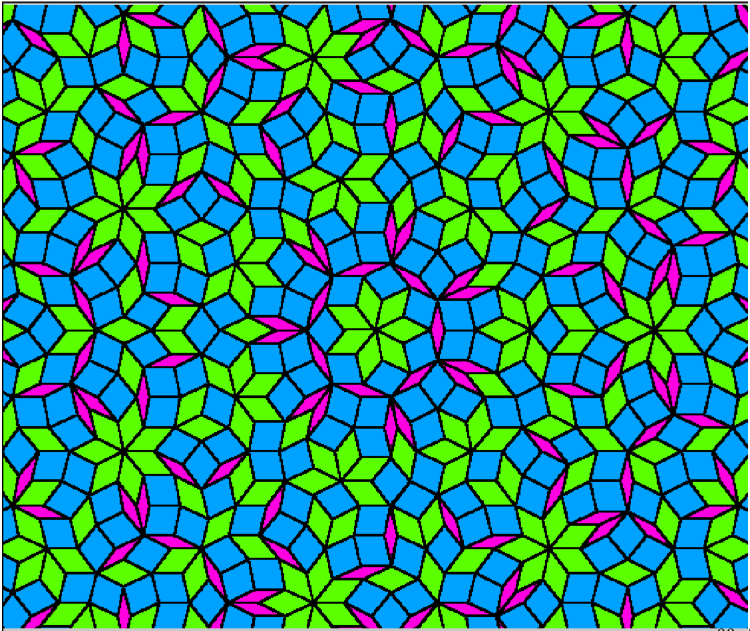
Cut and Project



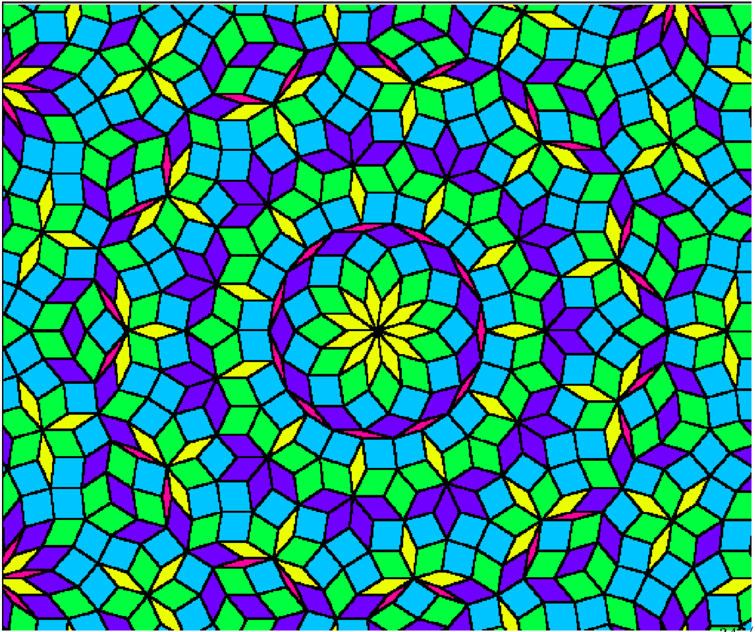
Cut And Project: 5-Fold (Penrose)



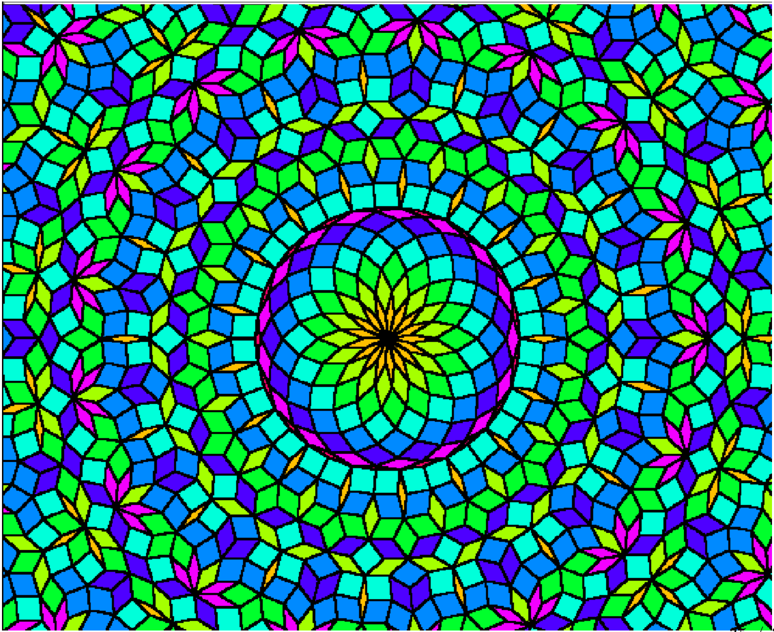
Cut And Project: 7-Fold



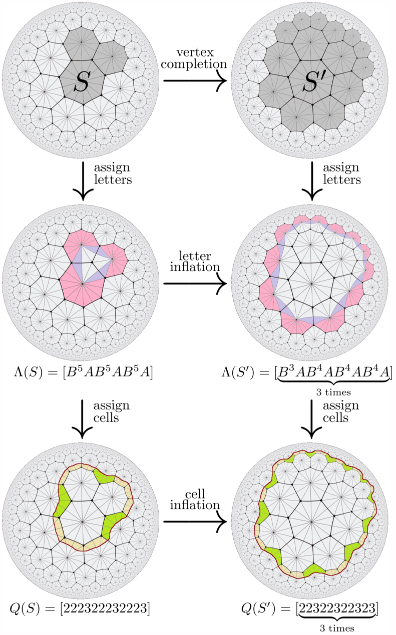
Cut And Project: 11-Fold



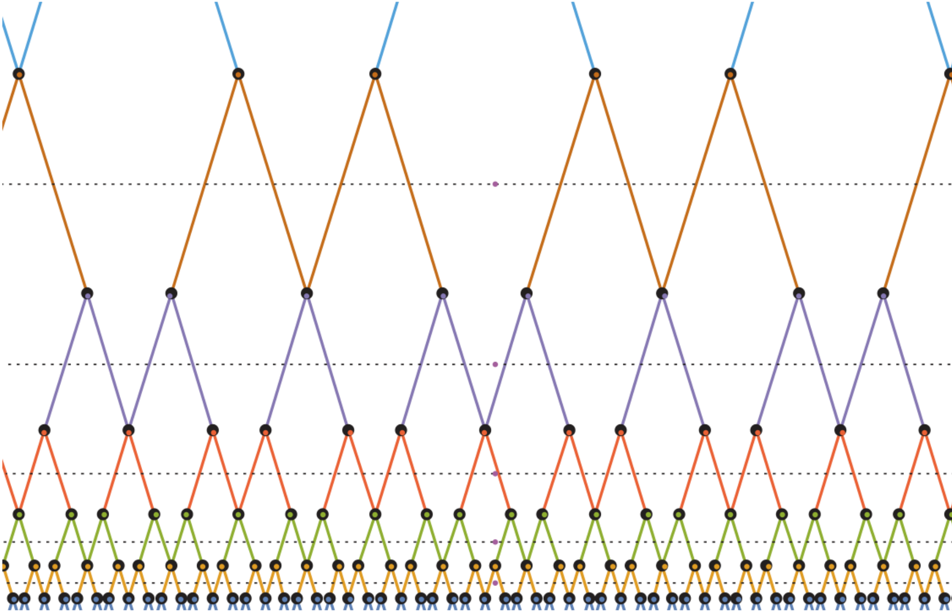
Cut And Project: 17-Fold



Other Junk: Holographic Quasicrystals



Other Junk: MERA



Other Junk: Topological Photonics

Topological Photonic Quasicrystals: Fractal Topological Spectrum and Protected Transport

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(Received 11 June 2015; revised manuscript received 24 September 2015; published 22 February 2016)

We show that it is possible to have a topological phase in two-dimensional quasicrystals without any magnetic field applied, but instead introducing an artificial gauge field via dynamic modulation. This topological quasicrystal exhibits scatter-free unidirectional edge states that are extended along the system's perimeter, contrary to the states of an ordinary quasicrystal system, which are characterized by power-law decay. We find that the spectrum of this Floquet topological quasicrystal exhibits a rich fractal (self-similar) structure of topological "minigaps," manifesting an entirely new phenomenon: fractal topological systems. These topological minigaps form only when the system size is sufficiently large because their gapless edge states penetrate deep into the bulk. Hence, the topological structure emerges as a function of the system size, contrary to periodic systems where the topological phase can be completely characterized by the unit cell. We demonstrate the existence of this topological phase both by using a topological index (Bott index) and by studying the unidirectional transport of the gapless edge states and its robustness in the presence of defects. Our specific model is a Penrose lattice of helical optical waveguides—a photonic Floquet quasicrystal; however, we expect this new topological quasicrystal phase to be universal.

DOI: 10.1103/PhysRevX.6.011016 Subject Areas: Optics, Photonics, Topological Insulators

Topological States and Adiabatic Pumping in Quasicrystals

Yaacov E. Kraus,[1] Yoav Lahini,[2] Zohar Ringel,[1] Mor Verbin,[2] and Oded Zilberberg[1]
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(Received 29 March 2012; published 4 September 2012)

The unrelated discoveries of quasicrystals and topological insulators have in turn challenged prevailing paradigms in condensed-matter physics. We find a surprising connection between quasicrystals and topological phases of matter: (i) quasicrystals exhibit nontrivial topological properties and (ii) these properties are attributed to dimensions higher than that of the quasicrystal. Specifically, we show, both theoretically and experimentally, that one-dimensional quasicrystals are assigned two-dimensional Chern numbers and, respectively, exhibit topologically protected boundary states equivalent to the edge states of a two-dimensional quantum Hall system. We harness the topological nature of these states to adiabatically pump light across the quasicrystal. We generalize our results to higher-dimensional systems and other topological indices. Hence, quasicrystals offer a new platform for the study of topological phases while their topology may better explain their surface properties.

DOI: 10.1103/PhysRevLett.109.106402 PACS numbers: 71.23.Ft, 05.30.Rt, 42.70.Qs, 73.43.Nq