Quasicrystalline Compactifications

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Strings: Point Particle Action

- The simplest Poincaré-invariant action for a point particle that is independent of parametrization is

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- We can add an additional field (a Lagrange multiplier) viewed as an independent world-line metric, $\gamma_{\tau\tau}(\tau)$, and define $\eta(\tau) = (-\gamma_{\tau\tau}(\tau))^{1/2}$, and get an equivalent action

$$S_P = \frac{1}{2} \int d\tau \left(\eta^{-1} \dot{X}^{\mu} \dot{X}_{\mu} - \eta m^2 \right) \tag{2}$$

  - Both Poincaré-invariant and reparametrization invariant

Strings: String Action

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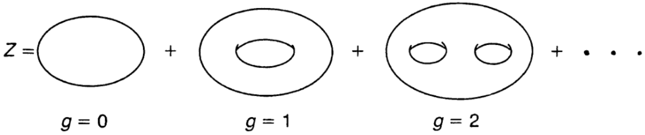
  - D-dimensional Poincaré-invariance
  - Reparametrization-invariance
  - Two-dimensional Weyl invariance

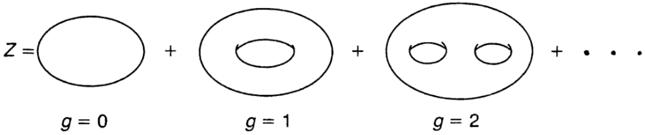
$$\begin{aligned} X'^\mu(\tau,\sigma) &= X^\mu(\tau,\sigma) \\ \gamma'_{ab}(\tau,\sigma) &= e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma) \end{aligned} \qquad (4)$$



- Naively, we have quantize by use of partition function

$$Z = \sum_{g=0}^{\infty} \int \mathcal{D}X \, \mathcal{D}\gamma \, e^{-S[X,\gamma]} \tag{5}$$

Strings: Critical Dimension



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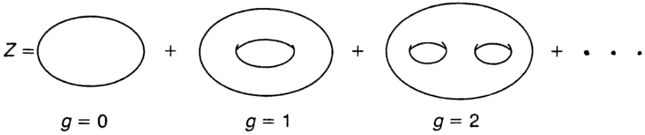

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- The measure is conformally invariant iff $D = 26$.

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- Assuming the string is closed, $X(\sigma,\tau) = X(\sigma + 2\pi, \tau)$ then the string coordinates are given by solutions to

$$(\partial_\sigma^2 - \partial_\tau^2)X^i = 0 \qquad i = 1, \ldots, d \tag{9}$$

Compactification: Lattice Conditions

- The solution to the wave equation can be decomposed into left and right moving solutions, $X^i = X^i_L + X^i_R$

$$X^i_L = x^i_L + p^i_L(\tau + \sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}^i_n}{n}e^{-2in(\tau+\sigma)} \qquad (10)$$

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- Consistency with the geometry enforces

$$(p_L, p_R) = (p/2 + w, p/2 - w) \tag{12}$$

  where $w \in \Lambda$ are winding numbers and $p \in \Lambda^*$ are the momenta.

  - The elements (p_L, p_R) are viewable as elements of an even unimodular lattice $\mathbb{II}_{d,d}$ with signature $(+^d, -^d)$.

Compactification: Orbifolds

- In a general toroidal compactification the fields form coordinates on

$$\mathbb{R}^{p,q}/\mathbb{II}_{p,q} \qquad \mathbb{II}_{p,q} \text{ even unimodular in } (p,q) \tag{13}$$

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- An orbifold is obtained by identifying elements not only if they are related by a shift a, but by some larger (discrete) automorphism group G of the lattice.
  - Elements are pairs (a, θ) where $\theta \in O(p) \times O(q) \subset O(p,q)$
  - The Hilbert space representation acts by

$$\theta\,|p_L, p_R\rangle = e^{2\pi i p\cdot a}\,|\theta_L p_L, \theta_R p_R\rangle \tag{14}$$

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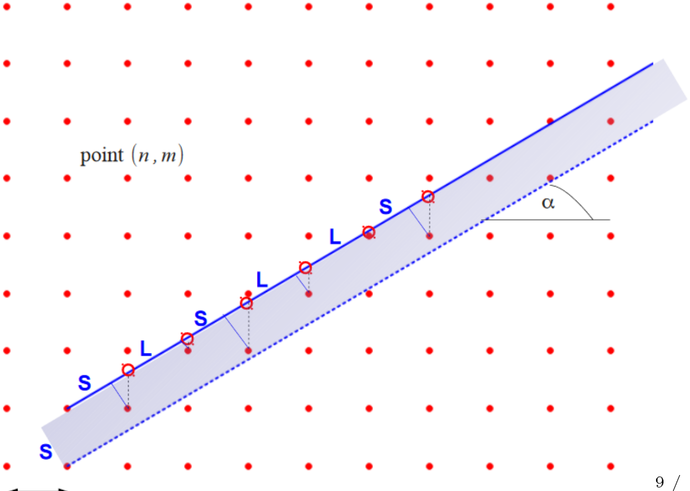
- If p_L and p_R belong to isomorphic spaces and $\theta_L = \theta_R$ we have a symmetric orbifold. Otherwise, we have an asymmetric orbifold.

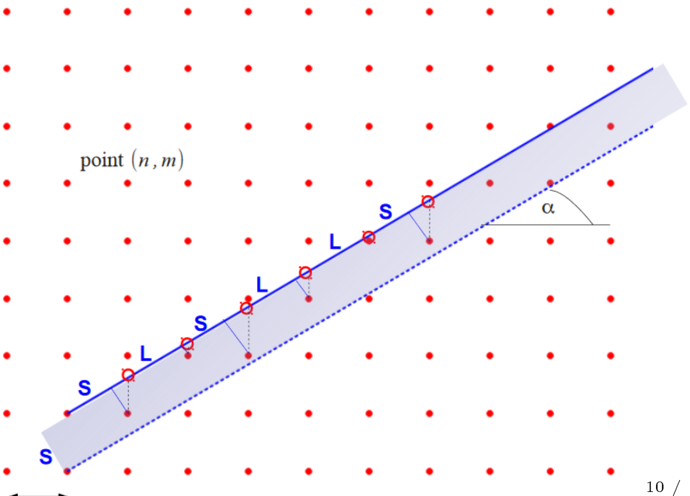
Asymmetric Orbifolds: Quasicrystals

- In constructions of quasicrystals, one starts with a regular lattice in a higher dimensional space, then tries to identify a fixed subspace which cuts the lattice at an irrational angle.

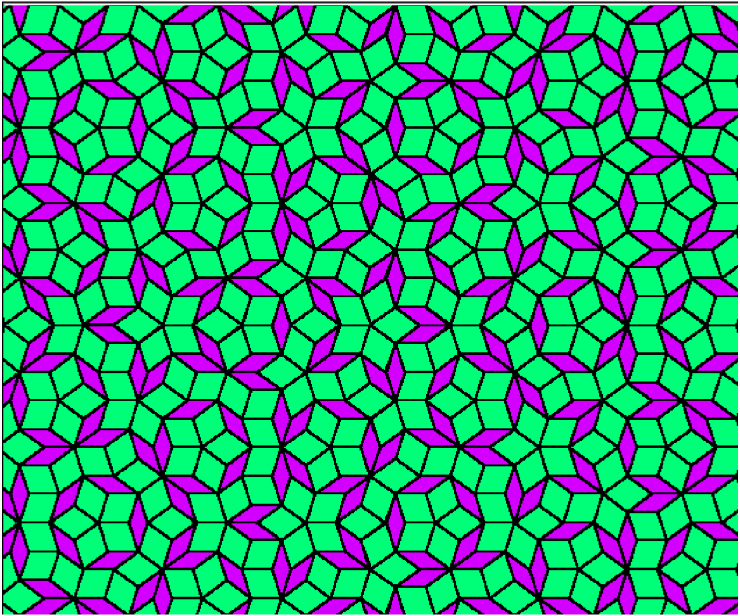


Asymmetric Orbifolds: Quasicrystals

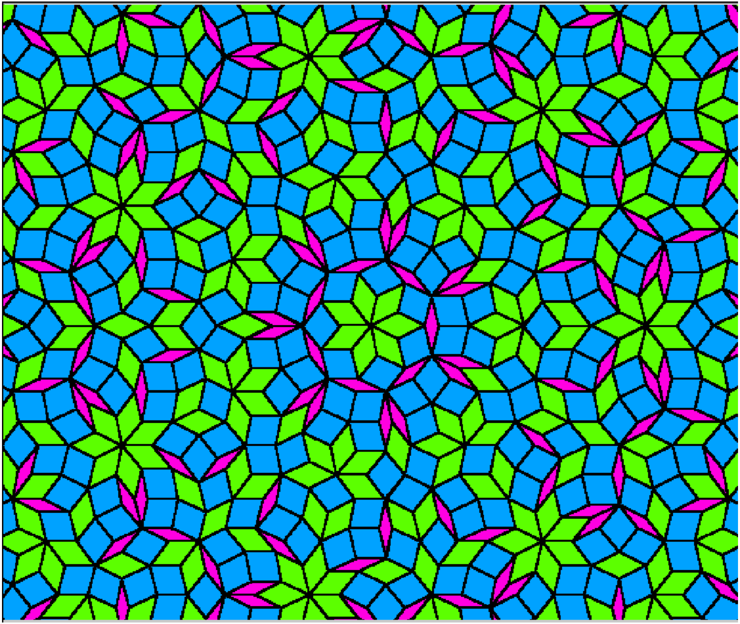
- Lattice within some acceptance range are projected onto the subspace forming a quasicrystal, which has symmetries forbidden in the projected dimension, but not the higher.



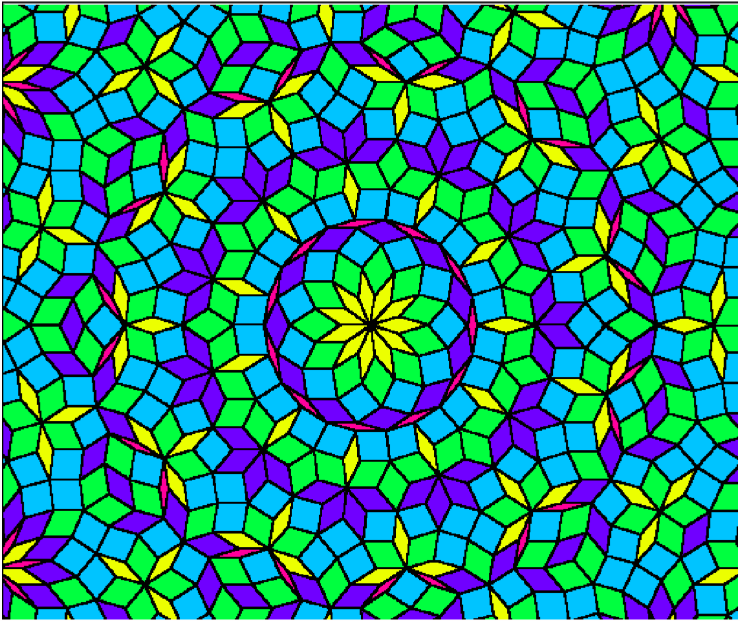
Asymmetric Orbifolds: Quasicrystals - 5 Fold



Asymmetric Orbifolds: Quasicrystals - 7 Fold



Asymmetric Orbifolds: Quasicrystals - 11 Fold



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- We will use the fact that $\mathbb{II}_{p,q}$ admits automorphisms that are not possible in p or q dimensions, to construct orbifolds with classically forbidden symmetries.

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- We will use the fact that $\mathrm{II}_{p,q}$ admits automorphisms that are not possible in p or q dimensions, to construct orbifolds with classically forbidden symmetries.
- Simplest quasicrystalline orbifolds allow one to project out most/all unwanted massless moduli (unwanted scalars).
- Resulting conformal field theory is not "close" to any rational conformal field theory
- Connections to string phenomenology, heterotic string, NAHE free-fermion models...