Orbifolds, Anomalies, and Topological Field Theories

for Perimeter Scholars International

Justin Kulp
supervised by Davide Gaiotto

Perimeter Institute for Theoretical Physics

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Introduction

Motivation

- Given a theory T with discrete abelian global G symmetry, what happens if we promote G to a local symmetry?
  - What does the new theory $T//G$ look like?
  - Ex. Gauging global $\mathbb{Z}_2$-symmetry of the high temperature Ising model produces low temperature Ising model. *Kramers-Wannier Duality*.
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- This operation is also known as *orbifolding T by G*.
- Can we even gauge a G symmetry in the first place?
  - Obstructions are so-called 't Hooft anomalies.
  - Intimately related to Wess-Zumino consistency conditions and Chern-Simons actions.

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  - Given a particular theory T we can generate a collection by orbifolding various subgroups $G \leq F$.
  - This collection does not depend on the specifics of T. Only G, F, and potential obstructions.

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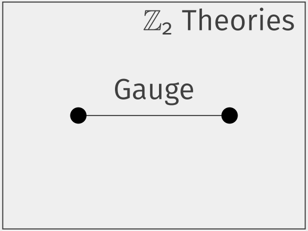

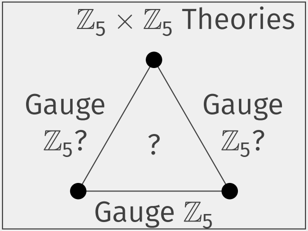

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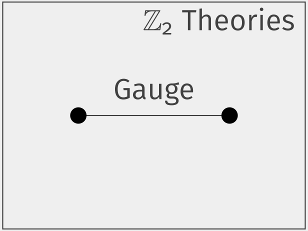

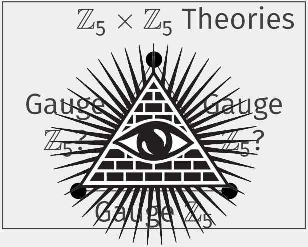

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Orbifolds and Anomalies

- The partition function for a CFT on a torus is

$$\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \tag{1}$$

- If we have abelian symmetry G, we restrict our space of states $\mathcal{H} \mapsto \mathcal{P}_G \mathcal{H}$ by projecting onto G-invariant states.
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- If we have abelian symmetry G, we restrict our space of states $\mathcal{H} \mapsto \mathcal{P}_G \mathcal{H}$ by projecting onto G-invariant states.
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- However, we must now trace over all *twisted sectors*, where the fields have "G-trivial" monodromy

$$\phi_{\text{tw.}} \left(e^{2\pi i} z, e^{-2\pi i} \bar{z} \right) = h \circ \phi_{\text{tw.}}(z, \bar{z}) \tag{2}$$

  - These twisted sectors must be included for a consistent CFT.

- Generally, for an orbifold with abelian symmetry G

$$\mathcal{Z} = \frac{1}{|G|} \sum_{g,h} \mathrm{Tr}_{\mathcal{H}_h} \left(g\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) =: \frac{1}{|G|} \sum_{g,h} \mathcal{Z}_{g,h} \qquad (3)$$

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- In language of gauging, we write $\mathcal{Z}_T[M,A]$ for the partition function of T with background G gauge field A. Then

$$\mathcal{Z}_{T//G}[M,B] \propto \sum_A e^{i(B,A)}\mathcal{Z}_T[M,A] \tag{4}$$

  is the partition function of the gauged theory, with background $\hat{G}$ gauge field B.
  - The gauged theory has a $\hat{G}$ symmetry which could be gauged next!

- An *'t Hooft anomaly* arises if we cannot make a global G-symmetry local and couple it to a gauge-field in a gauge invariant way.
  - When G is connected, Wess-Zumino conditions imply anomalies in d dimensions are classified by $d+1$ dimensional Chern-Simons actions. *Anomaly Inflow*.

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  - Dijkgraaf-Witten actions, classified by $H^{d+1}(G; U(1))$
- Result is we can only gauge a 2d theory with finite abelian symmetry G if a certain G-controlled cohomology class $[\omega] \in H^3(G; U(1))$ vanishes.
  - If it does then there are many inequivalent ways to gauge controlled by choice of $[\alpha] \in H^2(G; U(1))$.

SPT

- *Symmetry protected topological phases* are gapped phases of matter with G-symmetry which can be deformed to the trivial state without phase transition iff G-symmetry is broken.
  - $d+1$-dimensional SPT classified* by $H^{d+1}(G; U(1))$.
- The boundaries of SPT phases either break G-symmetry or carry an 't Hooft anomaly.
  - i.e. d-dim anomalous theory can be realized as a boundary for SPT phase in $d+1$.
  - If anomaly depends on d-dimensional fields, anomaly inflow cannot be cancelled by DW action. SET phase?

The Problem

The Operations

- Suppose we have a $2d$ theory T with $G = \mathbb{Z}_p \times \mathbb{Z}_p$ symmetry. We does the collection of orbifolds constructed from this theory look like?
- What are the operations we can perform on T?

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- What are the operations we can perform on *T*?
- Weak Transformations
  - "Change of Basis" for the group. Generators for automorphism group of *G*.

$$\pi_1 : \mathcal{Z}_{\alpha,\alpha';\beta,\beta'} \mapsto \mathcal{Z}_{\alpha,\alpha+\alpha';\beta,\beta+\beta'} \tag{5}$$

$$\pi_2 : \mathcal{Z}_{\alpha,\alpha';\beta,\beta'} \mapsto \mathcal{Z}_{\alpha',\alpha;\beta',\beta} \tag{6}$$

  - "Adding an SPT Phase" $\omega^{f(g,h)} \in H^2(G; U(1))$

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- Strong Transformations (Change Dynamics)
  - Gauge a $\mathbb{Z}_p$ subgroup

$$g : \mathcal{Z}_{\alpha,\alpha';\beta,\beta'} \mapsto \frac{1}{|G|} \sum_{\gamma,\delta} \omega^{\alpha\delta-\beta\gamma} \mathcal{Z}_{\gamma,\alpha';\delta,\beta'} \tag{8}$$

- Define the set of weak transformations $T_0 = \langle \pi_1, \pi_2, S \rangle$ and $T = \langle \pi_1, \pi_2, S, g \rangle$
- What are the groups T and T_0?

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    - Two theories will be identified if they are in the same T_0 orbit.
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- Computationally, we generate the T_0 and T groups and see if two cosets in T/T_0 share a common element modulo g-action.

First Attempt

- Treat a single twisted partition function $\mathcal{Z}_{\alpha,\alpha';\beta,\beta'}$ as a vector $|\alpha,\alpha';\beta,\beta'\rangle$ in a p^4 dimensional vector space over $\mathbb{Z}_p$.
- Our operations are $p^4 \times p^4$-dimensional matrices.

$$S\left|\alpha,\alpha';\beta,\beta'\right\rangle = \omega^{\alpha\beta'-\alpha'\beta}\left|\alpha,\alpha';\beta,\beta'\right\rangle \tag{9}$$

$$\pi_1\left|\alpha,\alpha';\beta,\beta'\right\rangle = \left|\alpha,\alpha+\alpha';\beta,\beta+\beta'\right\rangle \tag{10}$$

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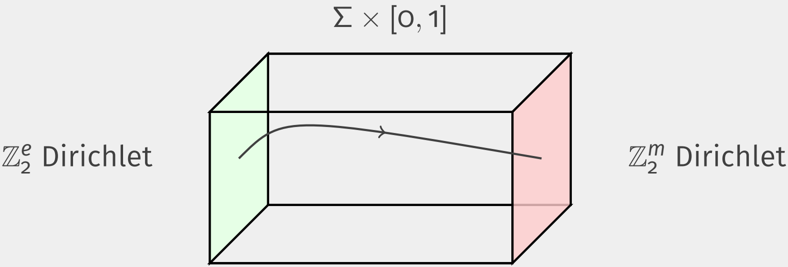
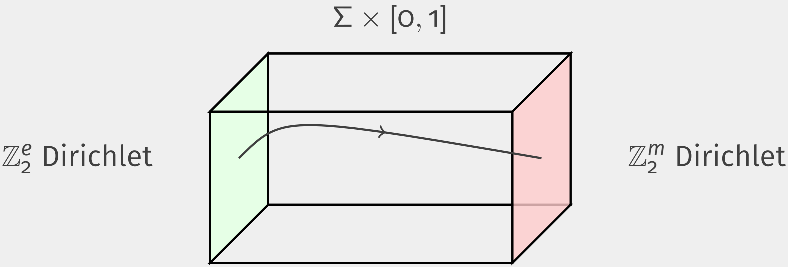
- Phases do not “see” the kets. The operators do not commute, but since p is prime every $|\alpha,\alpha';\beta,\beta'\rangle$ can be appended with ω^k for $0 \leq k < p$.

$$T_0 \cong \mathbb{Z}_p \times \mathrm{SL}_{\pm}(2,\mathbb{Z}_p) \tag{12}$$

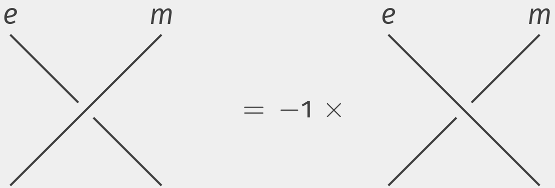
Topological Field Theory



- Consider $3d$ $\mathbb{Z}_2$ gauge theory on a slab $M = \Sigma \times [0, 1]$.



- Consider $3d$ $\mathbb{Z}_2$ gauge theory on a slab $M = \Sigma \times [0, 1]$.
- Two line operators, Wilson (e) and 't Hooft (m).



$2d$ Boundary Conditions

- $3d$ topological gauge theory on M associates $\mathcal{H}_{\Sigma}^{3d}$ to Σ.
  - A cycle $a \in H_1(\Sigma, \mathbb{Z}_2)$ can be labelled with a Wilson $L_e(a)$ or 't Hooft $L_m(a)$ line.
- Place Dirichlet $\mathbb{Z}_2^e$ boundary conditions Σ and Dirichlet $\mathbb{Z}_2^m$ boundary conditions (Neumann $\mathbb{Z}_2^e$) on $-\Sigma$.

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  - If Σ is equipped with Dirichlet $\mathbb{Z}_2^e$ boundary conditions given by $v \in H_1(\Sigma, \mathbb{Z}_2)$, this creates a state for the $3d$ theory $|e, v\rangle$.
  - Similarly for $-\Sigma$ with $|m, w\rangle$, the basis diagonalizing the 't Hooft magnetic line operators.

$$L_e(a)\,|e, v\rangle = (-1)^{\int a \wedge v}\,|e, v\rangle \tag{13}$$

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- We can couple our $2d$ theory T with $G = \mathbb{Z}_2^e$ symmetry to Σ, and gauge. The partition function can be read off

$$\sum_v \mathcal{Z}_T[\Sigma, v]\,\langle e, v | m, w\rangle = \sum_v e^{i(w,v)} \mathcal{Z}_T[\Sigma, v] = \mathcal{Z}_{T//\mathbb{Z}_2^e}[\Sigma, w] \tag{15}$$

The Solution

- We can interpret our 2*d* theories as boundary conditions for a 3*d* topological theory. In particular

$$ {}_T[0,1]_{\text{Dir}} = T \quad (16) $$

$$ {}_T[0,1]_{\text{Neu}} = T//G \quad (17) $$

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- Complicated math tells us: Symmetries of 3d TFTs are Brauer-Picard groups.
- Also, for a finite abelian group A

$$ \text{BrPic}(\text{Vec}_A) \cong O(A \oplus \hat{A}, q) \tag{18} $$

where $O(A \oplus \hat{A}, q)$ are automorphisms of $A \oplus \hat{A}$ preserving the canonical quadratic form.
  - In other words, $T = \langle g, S, \pi_1, \pi_2 \rangle \cong O(2, 2; \mathbb{Z}_p)$.

- This also gives more insightful basis. Obtained by simply Fourier transforming one of the copies of G in our basis

$$|(A,B),(C,D)\rangle_\star = \sum_{X,Y\in\mathbb{Z}_p} q((X,Y),(C,D))\,|A,B;X,Y\rangle \;. \qquad (19)$$

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- Now group operations send partition functions to partition functions. Not (weighted) linear combinations of them. They all act as 4×4 matrices on the indices now.

$$\begin{aligned} g\,|(A,B),(C,D)\rangle_\star &= |(C,B),(A,D)\rangle_\star & (20)\\ S\,|(A,B),(C,D)\rangle_\star &= |(B,A),(D,C)\rangle_\star & (21)\\ \pi_1\,|(A,B),(C,D)\rangle_\star &= |(A,A+B),(C-D,D)\rangle_\star & (22)\\ \pi_2\,|(A,B),(C,D)\rangle_\star &= |(A,B),(C-B,D+A)\rangle_\star\ . & (23)\end{aligned}$$

- This gives a more tractable basis for computation.

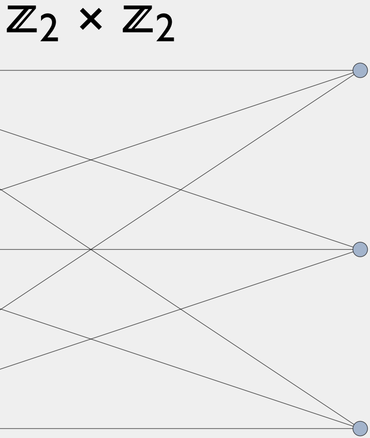

Figure: $|T_0| = 12$, $|T| = 72$. 6 orbifolds.

Some Plots: $\mathbb{Z}_3 \times \mathbb{Z}_3$

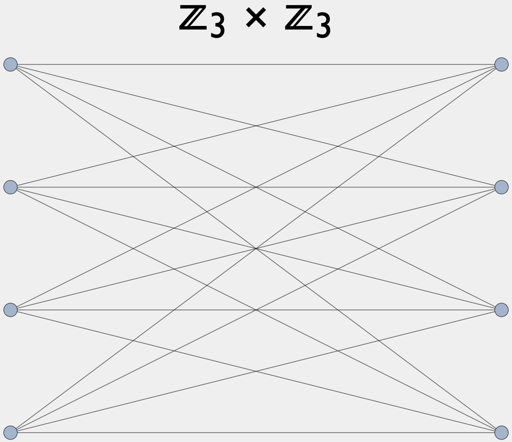

Figure: $|T_0| = 144$, $|T| = 1152$. 8 orbifolds.

Some Plots: $\mathbb{Z}_5 \times \mathbb{Z}_5$

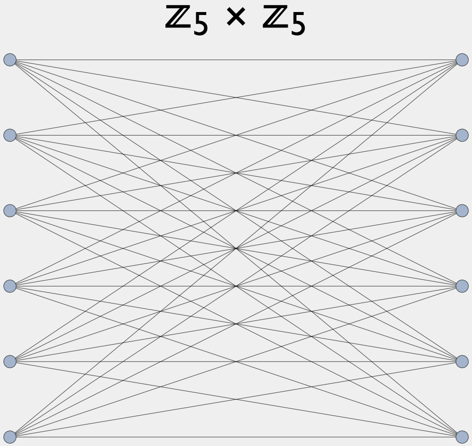

Figure: $|T_0| = 1200$, $|T| = 14,400$. 12 orbifolds.

Some Plots: $\mathbb{Z}_7 \times \mathbb{Z}_7$

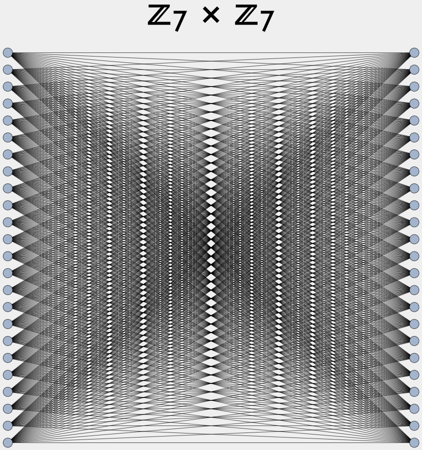

Figure: $|T_0| = 4704$, $|T| = 225,792$. 48 orbifolds.

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- Tried to understand the structure of orbifolds made from a theory.
  - Structure does not depend on theory, only on group and relevant anomalies.
- Introduced some notions from topological field theory to give a $3d$ interpretation to our $2d$ problem.
- Discovered for $G = \mathbb{Z}_p \times \mathbb{Z}_p$
  - $T_0 = \langle \pi_1, \pi_2, S \rangle \cong \mathbb{Z}_p \times \mathrm{SL}_{\pm}(2, \mathbb{Z}_p)$.
  - $T = O(2, 2; \mathbb{Z}_p)$.
  - The space of orbifolds forms a bipartite graph.

The next step (for next Monday!) is to extend this to theories of fermions.

- Don't know if there are results in the literature describing a way to find such a natural basis.
- SPT phases for fermions are described by "supercohomology" not regular group cohomology.
- Spin structures are affine $\mathbb{Z}_2$ gauge theories. Can add a $\mathbb{Z}_2$ connection to a spin structure but not vice-versa.
- Interesting connections to GSO projection, Jordan-Wigner transformation of Ising model, etc.

To be continued...

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- Fermions TBD.