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Orbifolds, Anomalies, and Topological Field Theories

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In this thesis we build general tools necessary to classify orbifolds of field theories with discrete abelian symmetries. We begin by defining an orbifold and elucidate how gauging a symmetry is equivalent to orbifolding. We investigate 't Hooft anomalies as potential obstructions to gauging and discuss their classification and link to SPT phases. With these tools we try to understand the structure of the space of $2d$ theories with $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry. We interpret our results in terms of a $3d$ topological gauge theory. We find that the structure of orbifolds of $2d$ theories with $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry has a simple representation in terms of bipartite graphs generated by some familiar matrix groups.

1 Introduction

Given a $2d$ quantum field theory T with discrete global symmetry G, there are a number of different manipulations one can perform on the theory, ranging from weak to very dramatic. One immediate task is to try to classify the structure of all theories with this symmetry and the operations that can carry us between them. In order to better understand this space we want to understand how such manipulations affect an arbitrary theory. Recently, classification programs of this sort have been enriched by developments in the mathematical and superconformal field theory literature [1, 2] and have connections to various topics in condensed matter research [3, 4]. As we will see, this also gives a way to understand the symmetries of abelian Dijkgraaf-Witten theories [5].

One example of a weak transformation is stacking the theory with a so-called SPT phase. Two gapped systems are in the same phase if they can be continuously deformed into one another. A gapped system is in the trivial phase if it can be continuously deformed to a trivial gapped system. An SPT phase with G-symmetry is a phase of matter that can be continuously deformed into the trivial state if and only if it breaks the G-symmetry [6].

An example of a strong manipulation on a theory is to gauge the abelian symmetry group G, thereby making it local. This is possible only if a certain G-controlled cohomology class $\omega \in H^3(G; U(1))$ vanishes. If it doesn't, the system has an 't Hooft anomaly. If it does, then there are many inequivalent ways to gauge a symmetry (because there are many inequivalent subgroups of G), and this is controlled by $\alpha \in H^2(G; U(1))$, with cohomologous α giving an equivalent gauge theory up to non-canonical isomorphism. This gauged theory could be written $T//_\alpha G$ [7]. We will find that, despite gauging the G-symmetry, the remaining theory still has a G-symmetry, or more precisely, a $\hat{G}$-symmetry where $\hat{G}$ is the Pontrayagin dual of G, meaning we are still in the same space of theories.

The purpose of this essay is to describe the generalities necessary for pursuing this classification program in a pedagogical manner, and then to solve the problem in the case $G = \mathbb{Z}_p \times \mathbb{Z}_p$ where p is prime. Along the way, we will develop full results for some simpler theories and uncover stepping stones to higher results. We will also see the remarkable fact that the structure of these operations do not depend very specifically on the theory we study, but only on the form of G and the anomalies that a G-symmetry theory can encounter.

Section 2 reviews the literature on CFT and expands our place for theories to live beyond the plane to arbitrary Riemann surfaces. We uncover modular invariance and the S and T-duality transformations. Then we look at gauging and see how gauging a symmetry is the same as orbifolding.

In Section 3 we clarify and generalize the notion of an anomaly from quantum field theory. We discover a very geometric way to present symmetries of a theory as networks of defects, and we look at how gauging affects this. From this presentation, cohomology will naturally arise in the discussion of SPT phases and anomalies, and we will see a direct link between the two subjects. Then we introduce the problem of understanding the space of theories in a rigorous way.

In Section 4 we introduce topological field theories and Wilson and 't Hooft lines. Then we describe how a $2d$ theory can act as boundary conditions for a $3d$ topological theory and interpret our problem in this new language. We use this formalism to find an answer to our problem. This section contains research which could be original in presentation for the physics community but seems to have been known in mathematical language for some time [2].

2 CFTs, Orbifolds, and Gauge Theories

In this section our goals will be three-fold: to introduce modular invariance and S and T transformations; to orbifold a theory with G-symmetry; and to show how orbifolding a theory has the same effect as gauging the symmetry. The machinery we establish in this section will provide concrete calculations for reference in the future, introduce twisted partition functions, as well as establish three of the four operations we described in the introduction: S and T transformations and gauging.

2.1 Some CFT Basics

A first study of CFT is typically conducted on $\mathbb{C}$ (or, up to a point, S^2). In this case, the holomorphic and anti-holomorphic components are completely independent, and often one can derive all the results on the holomorphic components and then copy them for the anti-holomorphic. The only interaction between the two sectors is when one "decomplexifies" and tries to go back to having continuous correlation functions; i.e. when $\bar{z}$ is literally the conjugate of z.

However, a CFT can be defined on any Riemann surface. In condensed matter these surfaces have applications in topological order, and in perturbative string-theory they are the higher loop scattering amplitudes. In QFT, one often finds stringent consistency conditions enforced by higher loop diagrams (like triangle diagrams), and one expects a similar restrictive phenomena between the holomorphic and anti-holomorphic sectors to arise here.

We will tackle the first of our goals for this section by warming-up with some CFT. In particular, we will investigate CFTs on the torus and the necessity of modular invariance, then we will study the free boson as an example. For the most part, we follow the relevant sections of [8, 9].

Modular Transformations

We construct a torus in two dimensions by taking a parallelogram and identifying opposite edges. Formally, we start with $\mathbb{C}$ and a lattice $\Lambda(\omega_1, \omega_2) = \{\omega_1 m + \omega_2 n \,|\, m, n \in \mathbb{Z}\}$, where $\omega_1, \omega_2 \in \mathbb{C}$. A torus is then homeomorphic to $\mathbb{C}/\Lambda(\omega_1, \omega_2)$ provided that $\omega_1 \neq \lambda\omega_2$ for $\lambda \in \mathbb{R}$.

However, despite two tori being homeomorphic, the complex structure inherited from $\mathbb{C}$ will be dependent on the choice of pair (ω_1, ω_2). It can be shown mathematically [10] that the pairs will define the same complex structure if they are related by a $\mathrm{PSL}(2, \mathbb{Z})$ transformation, but we can get there with physical reasoning.

First we note that we have an equivalence of lattices $\Lambda(\omega_1, \omega_2) = \Lambda(\omega_1', \omega_2')$ if the basis elements are related by an $\mathrm{SL}(2, \mathbb{Z})$ transformation. Now, the properties of a CFT on a torus should be scale and orientation independent, so only ratios like $\tau = \omega_2/\omega_1$ could be relevant. Our lattice identification

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{1}$$

acts on our relevant parameter, known as the *modular parameter*, by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,. \tag{2}$$

Clearly two matrices related by a -1 have no distinguishable action on τ, so we arrive at the following statement: *A torus is specified by one complex*

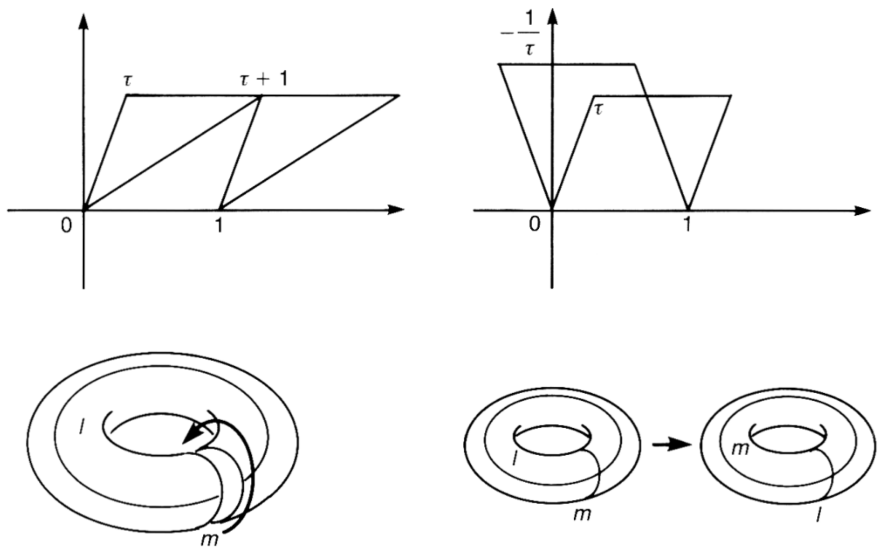


Figure 1: Left, a T-transformation puts a 2π-twist in a cylinder before closing it into a torus. Right, an S-transformation exchanges torus cycles. Image from [10].

number, the modular parameter τ, in the upper-half complex plane. Two tori are identified if they are related by a $PSL(2,\mathbb{Z})$ transformation.

It is a less simple task to show that $\mathrm{PSL}(2,\mathbb{Z})$ is generated by just two transformations:

$$T : \tau \mapsto \tau + 1 \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{3}$$

$$S : \tau \mapsto -\frac{1}{\tau} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

These transformations have straightforward geometric interpretations, see Figure 1.

Vacuum Energy on the Cylinder

We continue our CFT journey by recalling that $T(z)$ is not a primary field. Instead, under $z \mapsto f(z)$ it transforms as

$$T(z) \mapsto T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} S\left(f(z), z\right) \tag{5}$$

where $S(\cdot,\cdot)$ is the Schwarzian derivative.

In particular, for the plane to cylinder mapping, the transformation is $z = e^w$ where w are the coordinates on the cylinder. Evaluating, our stress-energy tensor transforms

$$T_{\text{cyl.}}(w) = z^2\,T(z) - \frac{c}{24}\,. \tag{6}$$

This shows the zero mode of the stress energy tensor is shifted on the cylinder

$$(L_{\text{cyl.}})_0 = L_0 - \frac{c}{24}\,. \tag{7}$$

Torus Partition Function

We construct our CFT on the torus by putting periodic boundary conditions along the traditional time-direction on the cylinder. On the plane, time increases radially outwards, so mapping into the cylinder means $\mathrm{Re}(w) = \mathrm{Re}(\log(z)) = \ln(r)$ is increasing. However, we will actually define $\mathrm{Re}(w)$ to be our spatial direction on the cylinder and $\mathrm{Im}(w)$ as time. This is fine since an S transformation exchanges these directions anyway.

Next suppose $\tau = \tau_1 + i\tau_2$. We note if we translate by τ_2 in time, that we've also translated τ_1 in space. So we define the partition function

$$\mathcal{Z}(\tau_1,\tau_2) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-2\pi\tau_2 H}e^{+2\pi\tau_1 P}\right)\,. \tag{8}$$

Since H generates times translation we can use our previous result about the vacuum energy on the cylinder to write

$$H_{\text{cyl.}} \approx -\frac{\partial}{\partial t} + E_0 = -\left(\partial_w + \partial_{\bar{w}}\right) - \frac{c+\bar{c}}{24}\,, \tag{9}$$

and

$$P_{\text{cyl.}} = i((L_{\text{cyl.}})_0 - (\bar{L}_{\text{cyl.}})_0)\,. \tag{10}$$

Converting to the planar L_0 and substituting into the partition function gives

$$\mathcal{Z}(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right)\,, \tag{11}$$

where $q = e^{2\pi i\tau}$.

Free Bosons on the Torus

We can calculate the free boson partition function really simply using the operator formalism[1]. The Laurent modes of $T(z)$ are the currents $j(z) = i\partial X(z)$, i.e.

$$L_0 = \frac{1}{2} j_0 j_0 + \sum_{k\geq 1} j_{-k} j_k \,, \tag{12}$$

so that the Fock space of oscillator modes is built up to be

$$|n_1, n_2, \ldots\rangle = j_{-1}^{n_1} j_{-2}^{n_2} \cdots |0\rangle \ . \tag{13}$$

The contribution of the oscillator modes is then

$$\begin{aligned}
\operatorname{Tr}\left(q^{L_0 - \frac{1}{24}}\right) &= q^{-\frac{1}{24}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left\langle n_1, n_2, \ldots \middle| \sum_{k=0}^{\infty} \frac{1}{k!} (2\pi i \tau)^k (L_0)^k \middle| n_1, n_2, \ldots \right\rangle \\
&= q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} q^{k n_k} \\
&= q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \frac{1}{1-q^k} \\
\operatorname{Tr}\left(q^{L_0 - \frac{1}{24}}\right) &= \frac{1}{\eta(\tau)}
\end{aligned} \tag{14}$$

where $\eta(\tau)$ is the Dedekind η-function. This leads us to

$$\mathcal{Z}'(\tau, \bar{\tau}) \stackrel{?}{=} \frac{1}{|\eta(\tau)|^2} \tag{15}$$

which is not modular invariant[2]. If we take the integral over the center-of-momentum of the system (the zero-mode contributions) we have

$$\mathcal{Z}_0(\tau, \bar{\tau}) \propto \int |q|^{p^2/2} \propto \frac{1}{\sqrt{\tau_2}} \,. \tag{16}$$

Multiplying gives us the full modular invariant partition function for the free-boson

$$\mathcal{Z}(\tau, \bar{\tau}) = \frac{1}{\sqrt{\tau_2}} \frac{1}{|\eta(\tau)|^2} \,. \tag{17}$$

[1] There is a method to derive partition functions through path-integral methods, but it involves zeta-regularization and horrifying calculations.

[2] See Appendix B.1 for details.

2.2 Orbifolds

Geometrically, an *orbifold* is a manifold M modulo the action of a discrete group G. Given a theory T with discrete symmetry group G, we will attempt to construct theories with an equivalence relation of fields and operators of the original theory modulo G, denoted $T//G$ [11]. In this section we will study orbifolding of theories and introduce the relationship to gauging a G-symmetry.

$\mathbb{Z}_2$ Orbifold of the Compactified Boson

In Appendix A.1 we construct the highest-weight states for a compactified boson on a circle. For our purposes, the important result is that they are $|m,n\rangle$, where $m,n \in \mathbb{Z}$. The $n \neq 0$ states correspond to the states winding around the circle, and $m \neq 0$ states correspond to momentum states with center-of-momentum m/R. Furthermore, the zero-mode of our compactified boson acts as

$$a_0 |m,n\rangle = \left(\frac{m}{R} + \frac{Rn}{2}\right) |m,n\rangle \ . \tag{18}$$

Following [8, 9], we will construct a $\mathbb{Z}_2$ orbifold of the compactified boson. Start with the field $\varphi(z)$ on the circle and impose a $\mathbb{Z}_2$ symmetry generated by $G : \varphi \mapsto -\varphi$. We see that G anticommutes with φ and all its mode operators individually,

$$G\varphi G^{-1} = -\varphi \, . \tag{19}$$

Now we see our original Hilbert space $\mathcal{H}$ will have to change from the original theory. In particular, we restrict to those states which are actually symmetric under the G-action by simply inserting the appropriate $\mathbb{Z}_2$ projector into our partition function

$$\mathcal{Z}_{\text{orbi.}}(\tau) = \text{Tr}_{\mathcal{H}}\left(\frac{1+G}{2} q^{L_0 - \frac{c}{24}}\right) = \frac{1}{2}\mathcal{Z}_{\text{circ.}}(\tau) + \frac{1}{2}\text{Tr}_{\mathcal{H}}\left(G q^{L_0 - \frac{c}{24}}\right) \ . \tag{20}$$

If we note that

$$a_0 G |m,n\rangle = G^2 a_0 G |m,n\rangle = -G a_0 |m,n\rangle = -\left(\frac{m}{R} + \frac{Rn}{2}\right) G |m,n\rangle \tag{21}$$

then we see that

$$G |m,n\rangle = |-m,-n\rangle \ , \tag{22}$$

and so only the states built on top of $|0,0\rangle$ will contribute (the symmetric and anti-symmetric states will give cancelling contributions), and will be included when we calculate the unknown piece of partition function

Using these results we can explicitly calculate the orbifold partition function, a straightforward calculation[3] similar to the free boson shows

$$\mathcal{Z}_{\text{orbi}}(\tau,\bar{\tau}) = \frac{1}{2}\mathcal{Z}_{\text{circ.}}(\tau,\bar{\tau}) + \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right|, \tag{23}$$

where we have re-included the anti-holomorphic piece now.

We immediately notice a problem: this partition function is not modular invariant. In particular, we see if we want to make it closed under modular transformations, we need to add a term

$$\mathcal{Z}_{\text{orbi.}}(\tau,\bar{\tau}) = \frac{1}{2}\mathcal{Z}_{\text{circ.}}(\tau,\bar{\tau}) + \left|\frac{\eta(\tau)}{\vartheta_2(\tau)}\right| + \underbrace{\left|\frac{\eta(\tau)}{\vartheta_3(\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_4(\tau)}\right|}_{\mathcal{Z}_{\text{tw.}(\tau,\bar{\tau})}}. \tag{24}$$

Investigating a single new (holomorphic) term we see

$$\sqrt{\frac{\eta(\tau)}{\vartheta_4(\tau)}} = q^{\frac{1}{16}-\frac{1}{24}}\prod_{n=0}^{\infty}\frac{1}{1-q^{n+\frac{1}{2}}}. \tag{25}$$

The $q^{\frac{1}{16}}$ gives away that we are dealing with a system with ground state energy 1/16, and the $q^{n+\frac{1}{2}}$ hints further that we are dealing with something anti-periodic. We now realize our error: while the original Hilbert space $\mathcal{H}$ should be restricted to its G-invariant states, a new *twisted Hilbert space* of states $\mathcal{H}_{\text{tw.}}$ can describe our system because certain anti-periodic states not in our original $\mathcal{H}$ are closed under the G-action. That is,

$$\mathcal{Z}_{\text{tw.}}(\tau,\bar{\tau}) = \text{Tr}_{\mathcal{H}_{\text{tw.}}}\left(\frac{1+G}{2}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right), \tag{26}$$

needs to be added to the total partition function.

[3] See the appendix for more special functions and their properties.

Orbifolds in General

Generally we see that the orbifold partition function is

$$\mathcal{Z}(\tau,\bar{\tau}) = \frac{1}{|G|}\sum_{g,h\in G} \mathrm{Tr}_{\mathcal{H}_g}\left(h\, q^{L_0}\bar{q}^{\bar{L}_0}\right) =: \frac{1}{|G|}\sum_{g,h\in G} \mathcal{Z}_{g,h}(\tau,\bar{\tau}). \tag{27}$$

h helps us project onto the G-invariant subspaces of the g-twisted sectors: those sectors whose states have "G-trivial" monodromy

$$\varphi_{\text{tw.}}\left(e^{2\pi i}z, e^{-2\pi i}\bar{z}\right) = g\,\varphi_{\text{tw.}}(z,\bar{z})\,g^{-1}\,. \tag{28}$$

We also can reconcile our previous error in a general principle: *including twisted sectors is necessary for modular invariance.* And further note that:

$$S : \mathcal{Z}_{g,h} \mapsto \mathcal{Z}_{h,g} \tag{29}$$

$$T : \mathcal{Z}_{g,h} \mapsto \mathcal{Z}_{g,g+h}\,. \tag{30}$$

This last statement is only true if so-called 't Hooft anomalies cancel. If not there will be phases as we will see in Section 3.

Gauging is Orbifolding

We end with an important note about the above, namely that gauging and orbifolding are the same operation [12].

Consider a $2d$ theory T with $\mathbb{Z}_2 = \langle g\rangle$ symmetry. We have an untwisted sector $\mathcal{H}_{T,\text{un.}}$ and a twisted sector $\mathcal{H}_{T,\text{tw.}}$, depending on whether or not this "background" $\mathbb{Z}_2$ is trivial or not. g splits these sectors based on if it acts as ± 1,

$$\mathcal{H}_{T,\text{un.}} = \mathcal{H}^{+}_{T,\text{un.}} \oplus \mathcal{H}^{-}_{T,\text{un.}} \tag{31}$$

$$\mathcal{H}_{T,\text{tw.}} = \mathcal{H}^{+}_{T,\text{tw.}} \oplus \mathcal{H}^{-}_{T,\text{tw.}}\,. \tag{32}$$

Now if we gauge this g, the gauge invariant states are

$$\mathcal{H}_{T//\mathbb{Z}_2} = \mathcal{H}^{+}_{T,\text{un.}} \oplus \mathcal{H}^{+}_{T,\text{tw.}}\,, \tag{33}$$

so this is like a new untwisted sector $\mathcal{H}_{T//\mathbb{Z}_2,\text{un.}}$, and the twisted sector would be $\mathcal{H}_{T//\mathbb{Z}_2,\text{tw.}} = \mathcal{H}^{-}_{T,\text{un.}} \oplus \mathcal{H}^{-}_{T,\text{tw.}}$.

If we define a new $\mathbb{Z}_2$ symmetry $= \langle \hat{g} \rangle$ that acts as $+1$ on elements from $\mathcal{H}_{T,\text{un.}}$, and -1 on $\mathcal{H}_{T,\text{tw.}}$, then this new $\hat{g}$ symmetry can be gauged to acquire the original theory, i.e. $T//\mathbb{Z}_2//\mathbb{Z}_2 = T$.

More abstractly [13], given a theory T, an abelian symmetry group G, and an irreducible representation of G, $\chi : G \to U(1)$ (necessarily one-dimensional as G is abelian); the theory $T//G$ has a symmetry which sends a twisted sector state from $\mathcal{H}_g$ to itself times $\chi(g)$. Since twisted sector states respect group multiplication, i.e. a state in $\mathcal{H}_{g_1}$ and $\mathcal{H}_{g_2}$ combine to form a state in $\mathcal{H}_{g_1 g_2}$, this χ symmetry is a well-defined symmetry of $T//G$. Since G is abelian the irreducible representations form a group, the Pontryagin dual $\hat{G} \cong G$, which is a symmetry of the new orbifolded theory obtained by assigning the appropriate charges to the twisted sectors. If we orbifold by this new $\hat{G}$, we end up with our original theory T

$$T//G//\hat{G} = T\,. \tag{34}$$

In the language of gauging [14, 15] we might write $\mathcal{Z}_T[M,A]$ for the partition function of T with background G-gauge field A, which is $H^1(M,G)$ valued. Then

$$\mathcal{Z}_{T//G}[M,B] \propto \sum_A e^{i(B,A)} \mathcal{Z}_T[M,A] \tag{35}$$

where B is the background field for the new theory, $H^1(M,\hat{G})$ valued, and (B,A) is the natural pairing:

$$e^{i(\cdot,\cdot)} : H^1(M,\hat{G}) \times H^1(M,G) \to H^2(M,U(1)) \cong U(1)\,. \tag{36}$$

That is, the partition function of $T//G$ is the discrete Fourier transform of T, so obviously $T//G//\hat{G}$.

We see that the twisted partition functions correspond to different possible G-connections of our theory (this is made explicit later), and by summing over all of them we are effectively making our connection dynamical.

3 Anomalies, Defects, and SPT Phases

In the following section our goals will again be three-fold. First we will review the notion of an anomaly and generalize their definition in a suitable way. Next, we will introduce a new way to view G-symmetries and G-bundles with flat connections as networks of defects, and interpret what it means to gauge a symmetry in this geometrical language. Following this, we will connect these two ideas whilst describing our final operation: adding an SPT phase. At the end we will introduce the main problem of study for this thesis.

3.1 Anomalies

We begin by reviewing some general statements about anomalies and try to figure out how best to capture the notion of an anomaly following [12, 16].

Noether's Theorem and Ward Identity

In field theory, Noether's Theorem associates a divergence-less current (and/or a conserved charge) to every continuous symmetry of the system, and vice-versa [17]. Namely, if

$$\begin{aligned} \phi(x) &\to \phi'(x) = \phi(x) + \epsilon\, \delta\phi(x) \\ \mathcal{L}(x) &\to \mathcal{L}(x) + \epsilon\, \partial_\mu \mathcal{J}^\mu(x)\,, \end{aligned} \tag{37}$$

then, on-shell, the current

$$J^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - \mathcal{J}^\mu \tag{38}$$

is divergence free.

In the path-integral formalism, we instead start with a functional integral coupled to a classical source $K(x)$

$$Z[K] = \int \mathcal{D}\phi\, \exp(-S[\phi] + K\cdot\phi)\,. \tag{39}$$

This sees a symmetry transformation as a change of variables and so is left unchanged

$$Z[K] = \int \mathcal{D}\phi'\, \exp(-S[\phi'] + K\cdot\phi')\,. \tag{40}$$

If we expand the fields, straightforward manipulations lead us to

$$Z[K] = \int \mathcal{D}\phi'\, \exp(-S[\phi] + K\cdot\phi)\left(1 - \int \mathrm{d}^d x\, \epsilon(\partial_\mu J^\mu - K\delta\phi) + \dots\right). \tag{41}$$

Now, if the new measure is the same as the old measure, then by the fact that the partition functions are identical the new additions must be exactly 0, giving the Ward Identity for the current after taking functional derivatives and setting $K = 0$ as we do:

$$\partial_\mu \langle J^\mu(x)\phi(x_1)\cdots\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle\phi(x_1)\cdots\delta\phi(x_i)\cdots\phi(x_n)\rangle\,. \quad (42)$$

A Strong Assumption

One assumption we made was that the new measure after transformation is the same as the old measure. This is way too strong of an assumption. When this is not true an anomaly arises.

Often people say that a symmetry is classically obeyed but quantum mechanically violated. This is a fallacy that comes from going backwards from classical physics to quantum mechanics: in the quantum theory the symmetry doesn't exist; it may look like a symmetry exists because it is present in the Lagrangian or appears at a low energy [18]. Moreover, the Lagrangian is not the defining object for our quantum mechanical theory, that object is the partition function, so we will seek a more partition function-centric definition[4]. Indeed, when we write the partition function we separate it into a "sum over histories" and a Lagrangian

$$Z[K] = \underbrace{\int \mathcal{D}\phi}_{\text{sum over histories}} \quad \underbrace{e^{-\int \mathrm{d}^d x\,\mathcal{L}}}_{\text{Lagrangian}}\,. \quad (43)$$

We will still talk about things as if there is a breaking of classical symmetry, but we will also keep in mind that this is not strictly correct.

Anomalies come in three forms:

1. *Global Anomalies* arise when passing from a classical theory with a global symmetry to a quantum one without that symmetry. These are generically fine, but lead to some possible detectable observables by changing the dynamics of the theory.

[4] In a first QFT class one often starts with a classical theory and then quantizes it, and one must immediately suspect that there could be theories which don't come from any classical Lagrangian, and indeed this is the case [19].

2. *Gauge Anomalies* arise when a gauge symmetry is not kept when quantizing. These are terrible. Such a situation leads to negative-norm states and non gauge-invariant observables.
3. *'t Hooft Anomalies* arise when a symmetry is consistent globally, but the gauged symmetry is anomalous [20].

Example: The Chiral Anomaly in $d = 2$

The *chiral* or *axial* or *ABJ* anomalies are the canonical example of global anomalies, which first arose in the study of $\pi^0 \to \gamma\gamma$ decays [21, 22]. Consider a massless non-chiral fermion on $\mathbb{R} \times S^1$ with Dirac-algebra

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \quad \gamma^3 = \gamma^0\gamma^1 = \sigma^3\,, \tag{44}$$

coupled to a background $U(1)$ field

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi\,. \tag{45}$$

We find two symmetries of the Dirac fermions:

$$\text{Vector:} \quad \psi \to e^{i\alpha}\psi, \qquad J^\mu = \bar{\psi}\gamma^\mu\psi \tag{46}$$

$$\text{Axial:} \quad \psi \to e^{i\alpha\gamma^3}\psi, \quad J_3^\mu = \bar{\psi}\gamma^\mu\gamma^3\psi\,. \tag{47}$$

Or, combining these results in a linearly independent way, our chiral fermions satisfy the current conservation equation

$$\partial_\mu\bar{\psi}_\pm\gamma^\mu\psi_\pm = 0\,, \tag{48}$$

which leads to separate classical conservation of the left and right-moving fermions, n_- and n_+, or *chiral symmetry*. This is not correct precisely due to our background field.

We can expand our gauge field on our circle of length L as

$$A_\mu(t,x) = \sum a_\mu(t)\cos\left(\frac{2\pi n}{L}x\right)\,. \tag{49}$$

We can make A_1 constant in space with a gauge transformation. Then we can make another gauge transformation, adding $\partial_\mu\alpha$ of $\alpha = 2\pi nx/L$, which doesn't harm the periodicity of A_μ. The result is that $A_1 \sim A_1 + 2\pi n/L$, so that the A_1 field lives on a circle of radius $1/L$.

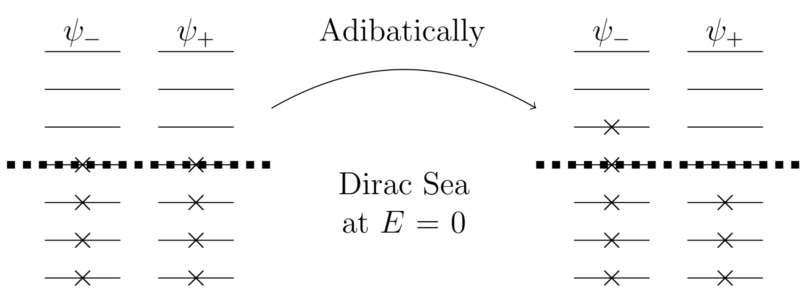
Now suppose we have a particular background field setup so that $A_0 = 0$, and A_1 is varying adiabatically. Then the Dirac equation is equivalent to

$$\left[\partial_0 + \gamma^5(\partial_1 - iA_1)\right]\psi = 0\,. \tag{50}$$

A Fourier expansion of ψ implies the use of ansatz $\psi = e^{-iE_n t}e^{\frac{2\pi i n}{L}x}$, which we can plug in to the Dirac equation to find

$$E_n^{\pm} = \pm\frac{2\pi n}{L} \mp A_1 \tag{51}$$

where $E_n^{\pm}$ refers to the energy of the left and right moving components; thus our setup has two towers of fermion energy levels.



When we adiabatically change A_1 by an amount $2\pi/L$, the energy spectrum goes back to its original form, but each ψ_+ is shifted down one level in the ladder and each ψ_- is shifted up one level: a right-moving fermion disappears and a left moving fermion appears. The axial charge, $\Delta Q_3 = n_+ - n_-$, shifts by -2, and so we see

$$\Delta Q_3 = -\frac{L}{\pi}\Delta A_1\,, \tag{52}$$

from which we may go back to the current form to get

$$\partial_\mu J_3^\mu = \frac{1}{\pi}\epsilon^{\mu\nu}F_{\mu\nu}\,. \tag{53}$$

The fact that we were able to do this is related to the fact that the Dirac sea is an infinitely filled well of lower energy states, hence we can steal from ψ_+ and add to ψ_- without changing the number of right-moving states. This non-truncation of the Dirac sea is an indication that this phenomena is linked to regularization, and the so-called *Fujikawa viewpoint* makes this explicit by showing that the measure transforms

$$\psi \mapsto e^{i\alpha\gamma^3}\psi \tag{54}$$

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \mapsto \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[2i\alpha \operatorname{Tr}\gamma^3\right] , \tag{55}$$

where Tr is the *regularized trace*. The result is

$$\operatorname{Tr}\gamma^3 = \frac{1}{\pi}\int d^2x\, \epsilon_{\mu\nu}F^{\mu\nu} , \tag{56}$$

so that our theory has an $\alpha \in U(1)$ controllable phase ambiguity on the partition function.

We can fix our philosophical and pragmatic issues with the definition of an anomaly by generalizing: *A system with symmetry G has an anomaly if the partition function has a G-controllable phase ambiguity* [12].

3.2 Defects and Connections

We now move on to explaining a new and useful way to view G-bundles with flat connections, which is particularly useful for understanding gauging of discrete symmetry groups. This approach will also naturally lead us to cohomological classifications and interpretations of anomaly inflow when we study SPT phases.

Defects as Symmetries

Suppose we study a theory with a continuous symmetry described by group G, this leads to a conserved current j by Noether's theorem and/or a conserved charge Q. We obtain this charge by integrating the current over a codimension 1 subspace, $M^{(d-1)} \subset M$, of our spacetime

$$Q(M^{(d-1)}) = \int_{M^{(d-1)}} j\, . \tag{57}$$

We typically take $M^{(d-1)}$ to be space at a fixed time-slice (typically $M = M^{(d-1)} \times \mathbb{R}$), but any closed codimension 1 subspace partitioning the spacetime into two pieces will suffice.

We also know that, given a conserved charge, this charge acts as the infinitesimal generator for our symmetry. In particular, a transformation implementing the action of $g \in G$ on the system will be described by

$$U_g(M^{(d-1)}) \sim \exp(i\omega Q)\,. \tag{58}$$

Where i and ω are whatever appropriate weights are needed to specify the group element g.

More broadly, for any symmetry group, discrete or continuous, we can define $U_g(M^{(d-1)})$ by cutting the spacetime along $M^{(d-1)}$ and inserting a group transformation in the complete set of states for the Hilbert space associated to that transformation. This introduces a discontinuity of the fields across $M^{(d-1)}$ by applying the g-transformation [15].

If instead of $M^{(d-1)}$ we associate an operator with an open space $\Sigma^{(d-1)} \subset M$, then we can also add boundary operators $\gamma^{(d-2)} = \partial\Sigma^{(d-1)}$. If we imagine coupling the system to background G-gauge fields, then the operator associated with $\Sigma^{(d-1)} \cup \gamma^{(d-2)}$ is flat up to g-monodromy around $\gamma^{(d-2)}$ [15].

Connections as Networks

From the insight above, we can view a background flat gauge connection as a network of defects. The defects partition the manifold into open sets equipped with transition functions $g_{\alpha\beta}$ on the intersection of two sets $U_\alpha \cap U_\beta$, and agreeing on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$. If one is familiar with characteristic classes and transition functions, one sees cohomology will arise to describe these intersections in a natural way [10, 23]. We will show this in a simpler manner later on.

We also claim that all discrete gauge connections are necessarily flat. Mathematically, this is true because the space of all G bundles over M, up to isomorphism, are in bijection, up to homotopy class, with maps from $M \to BG$ [24]. Furthermore, the space of all G bundles with flat connection is specified by G-monodromy on M, i.e. maps $\pi_1(M) \to G$ [25]. These are equivalent in the case of G-finite. Alternatively, we could just say in a discrete gauge theory we can't apply incremental transformations.

In summary: *A background flat gauge connection for a G symmetry on M can be represented by a network of codimension 1 defects that apply the transformation $g \in G$ when an object passes through the defect. For discrete gauge groups, every connection can be represented this way.* See Figure 2.

If we make the finite G-symmetry above dynamical (we gauge the theory), then defects apply a gauge transformation and hence all act trivially in the new theory. The closed defects all disappear and the defects represented by open sets disappear and only their boundary terms $\gamma^{(d-2)}$ survive. These boundary terms now act as operators associated with the twisted sectors of the gauge theory (remember they implemented G-trivial monodromy terms). To make the theory dynamical we integrate over all gauge field values, or for us, sum over all flat connections and/or networks of defects [15].

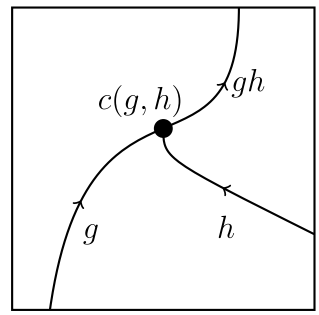


Figure 2: g and h lines combine to form gh. A phase is added at each trivalent junction.

3.3 SPT Phases, Cohomology, and Anomaly Inflow

In this section we will unify our discussion of anomalies with our network representations for G-bundles with flat connection. First we will introduce the notion of discrete torsion for orbifolds as considered by Vafa [26], and then explain how this is connected to the more general and emerging study of SPT phases. This will lead us naturally to a discussion of the classification of SPT phases and group cohomology, as well as the relationship to anomalies through anomaly inflow.

Discrete Torsion

We can now elaborate on our comment about 't Hooft anomalies following Equations (29) and (30). Suppose we actually try to gauge a $d = 2$ theory with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. If $\mathcal{Z}$ is the original partition function there are inequivalent ways to gauge a $\mathbb{Z}_2$ subgroup because there are many $\mathbb{Z}_2$ symmetries in $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, if we try to write a partition function, one finds that there is a sign ambiguity, i.e. while

$$\mathcal{Z} = \sum_{\substack{(g,h),(g',h') \\ \in \mathbb{Z}_2 \times \mathbb{Z}_2}} Z_{(g,h),(g',h')} \tag{59}$$

is modular invariant, so too is the linear combination

$$\mathcal{Z} = \sum_{\substack{(g,h),(g',h') \\ \in \mathbb{Z}_2 \times \mathbb{Z}_2}} (-1)^{gh'-hg'} Z_{(g,h),(g',h')} \,. \tag{60}$$

So there is a phase ambiguity $\epsilon(g,h)$ on some of our twisted partition functions, known as *discrete torsion* [26, 27]. It was first discovered by Vafa in the context of string partition functions and has interesting implications for the study of Calabi-Yau manifolds and their singularities.

By assigning a phase factor to a twisted sector, we are assigning a phase to a homotopy class of strings $\pi_1(\Sigma) \to G$ [28], which we recognize as assigning a phase to G-bundles with flat connection (up to isomorphism) from the above discussion. This further elucidates how orbifolding is gauging.

Now we ask: how free are we to pick discrete torsion on a Riemann surface Σ? First note that a twisted partition function $\mathcal{Z}_{g,h}$ on the torus is specified by two group elements $(g,h) \in G^2$ stating how the states in that sector close around the cycles of the torus. Similary, taking this description to a $g = 2$ surface, which has 4 independent cycles $(b_1, a_1; b_2, a_2)$ (see Figure 3), a twisted sector is identified by choosing $(g_1, h_1; g_2, h_2) \in G^4$. We proceed by pulling the two holes in the surface apart so that we have two tori connected by a point-like tube and a very long c-cycle twisting them together. If we assume our assignment of phases is consistent with one-loop calculations, and this pulling-factorization method, then the only new cycle to consider is this c-cycle. A Dehn twist around c acts to send $(g_1, h_1; g_2, h_2) \mapsto (g_1 h_2 h_1^{-1}, h_1; g_2 h_1 h_2^{-1}, h_2)$ [26].

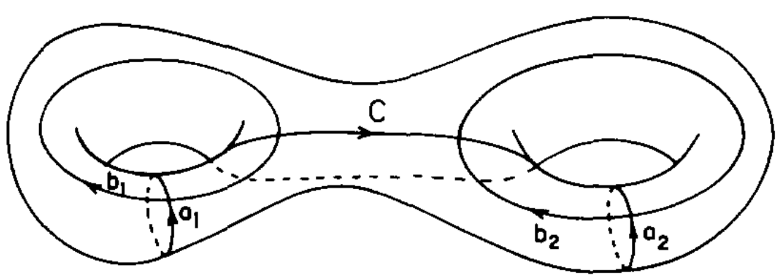


Figure 3: A genus 2 Riemann surface, $c = a_1^{-1} a_2$. A genus g Riemann surface has $2g$ independent cycles. Image taken from [26].

The result is that any phases must satisfy:

$\epsilon(1,1) = 1$	Normalization,
$\epsilon(g,h) = \epsilon(g^a h^b; g^c h^d)$	1-loop modular invariance,
$\epsilon(g_1,h_1;g_2,h_2) = \epsilon(g_1 h_2 h_1^{-1}, h_1; g_2 h_1 h_2^{-1}, h_2)$	Higher modular invariance,
$\epsilon(g_1,h_1;g_2,h_2) = \epsilon(g_1,h_1)\epsilon(g_2,h_2)$	Factorizability.

Which Vafa shows in [26] is equivalent to

$$\epsilon(g,g) = 1 \tag{61}$$
$$\epsilon(g,h) = \epsilon(h,g)^{-1} \tag{62}$$
$$\epsilon(g_1 g_2, g_3) = \epsilon(g_1,g_3)\epsilon(g_2,g_3)\,. \tag{63}$$

The result is then: *a theory can have discrete torsion iff* $H^2(G,U(1)) \neq 0$. This historical view relies heavily on a stringy viewpoint and restricting calculations with higher loop consistency conditions. This isn't necessary as we will see[5].

SPT Phases

From Section 2 we can convince ourselves that these choices of phases are weightings of the flat-connections in a sum over flat connections. We can view these phases $\epsilon(g_1,g_2) \sim e^{iS}$, so that an individual phase gives a partition function for a non-dynamical theory T_G^α with global symmetry group G, controlled by the so called *Dijkgraaf-Witten action* $S_\alpha[A]$, where $\alpha \in H^2(G;U(1))$, and A is a flat G-connection [15].

These theories T_G^α define *symmetry protected topological phases* (SPT phases): gapped phases of matter with a G-symmetry which can be smoothly deformed to the trivial state if and only if the G symmetry is broken [3]. It was shown in [29] that every SPT phase with G-symmetry can be described with a topological action, encoding its coupling to a spacetime and flat background G-connection.

The discrete torsion theories of Vafa no longer arise by mysteriously adding phases to an orbifolded CFT, but by orbifolding the theory with an SPT phase coupled to it CFT $\times\, T_G^\alpha$.

[5]Or as put: "No torus was harmed [in the making of this calculation]."

Cohomology in Defect Networks

How does this arise in the language of symmetry-implementing defects? Consider a $2d$ space with a background G-bundle network. We assign a partition function[6], or *Dijkgraaf-Witten action*, to this theory by

$$\mathcal{Z} = \prod_{\substack{\text{All}\\ \text{Junctions}}} c(g,h) \tag{64}$$

where $c(g,h)$ is a $U(1)$ phase assigned to a trivalent junction with an incoming g and h defect as in Figure 2.

A single G-bundle can be expressed in equivalent ways as a network of domain walls (see Figure 4). So demanding gauge invariance of $\mathcal{Z}$ enforces the condition that

$$c(g,h)c(gh,k) = c(g,hk)c(h,k)\,. \tag{65}$$

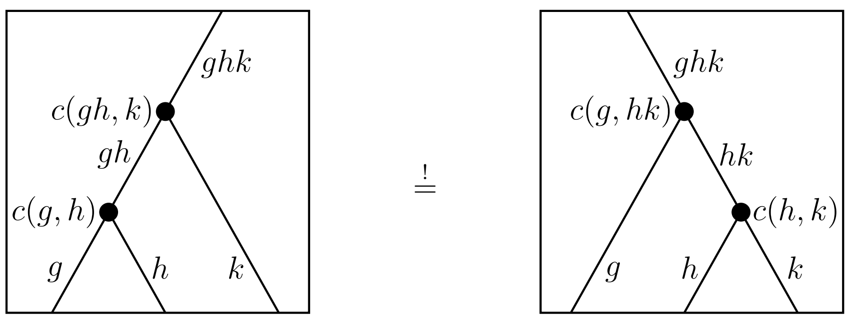


Figure 4: We enforce that these presentations of the G-bundle be identified.

These crossing-symmetry style relations are what imply we don't need to go to higher genus surfaces to get consistency conditions.

Furthermore, if we redefined our phases so that each g line carried an additional inbound $b(g)$ or outbound $b(g)^{-1}$, then under this redefinition

$$c(g,h) \to \tilde{c}(g,h) = c(g,h)\frac{b(g)b(h)}{b(gh)}\,, \tag{66}$$

[6]This was considered by Dijkgraaf and Witten in [28] in the context of dynamical G gauge fields. They also show in a straightforward homological way that this is really *the* action for describing such a finite group gauge theory.

and so our phase assignments don't matter up to this equivalence.

This means our distinct actions are characterized by

$$\frac{c(g,h)\,\text{satisfying Eq.65}}{c(g,h)\,\text{of the form Eq.66}} = H^2(G;U(1))\,. \tag{67}$$

In higher dimensions we can repeat this procedure to find: *defect networks in d+1 dimensions are classified by* $H^{d+1}(G;U(1))$. Or, alternatively: *bosonic SPT phases in* $d+1$ *dimensions are described by Dijkgraaf-Witten actions controlled by* $\alpha \in H^{d+1}(G;U(1))$ [12, 15].

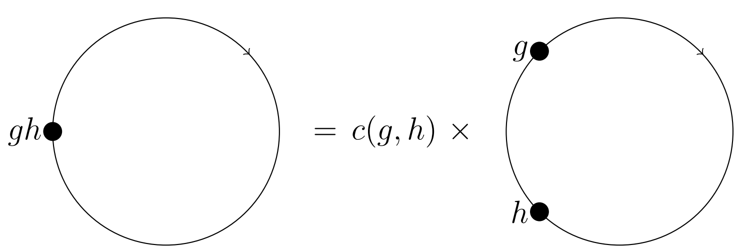
Cohomology in Quantum Mechanics

Group cohomology and anomalies appear in even the simplest system we can consider [12]. Consider a $(0+1)d$ QFT, aka. quantum mechanics, with finite dimensional Hilbert-space, finite symmetry group G, and Hamiltonian $H=0$.

As explained in the review article [30], quantum mechanical states are actually unit rays of the Hilbert space, so $\psi \in \mathcal{H}$ and $c\psi \in \mathcal{H}$ are equivalent if $c \in U(1)$. Consequentially, the G-symmetry acts on $\mathcal{H}$ projectively. That is, for $g,h \in G$ and representation $\rho : G \to GL(\mathcal{H})$, the representation is only required to satisfy

$$\rho(g)\rho(h) = c(g,h)\rho(gh)\,, \quad c(g,h) \in U(1)\,. \tag{68}$$

Now suppose we periodicitize time on S^1 so we can write the partition function. Then insert the transformation gh, so that $\mathcal{Z}(gh) = \mathrm{Tr}\left(e^{-\beta H}\rho(gh)\right) = \mathrm{Tr}\,(\rho(gh))$. We can represent this transformation another way though, by splitting g and h, so that $\mathcal{Z}(gh) = c(g,h)\,\mathrm{Tr}\,(\rho(g)\rho(h))$.



Our partition function has a G-controllable phase anomaly from $c(g,h)$.

We can't simply assign any phase however; these anomalies are constrained. Imagine one inserted three operators $g, h, k \in G$. Then we could split them in two different ways:

$$\rho(ghk) = \rho(gh)\rho(k)c(gh,k) = \rho(g)\rho(h)\rho(k)c(g,h)c(gh,k) \tag{69}$$

$$\rho(ghk) = \rho(g)\rho(hk)c(g,hk) = \rho(g)\rho(h)\rho(k)c(c,hk)c(h,k)\,, \tag{70}$$

so we must have that

$$c(g,h)c(gh,k) = c(g,hk)c(h,k)\,. \tag{71}$$

Furthermore, we could redefine our $U(1)$ phases, i.e. we could define $\tilde{\rho}(g) = b(g)\rho(g)$ for $b(g) \in U(1)$, which leads to

$$\tilde{c}(g,h) = c(g,h)b(gh)b^{-1}(g)b^{-1}(h)\,. \tag{72}$$

Thus the set of eligible projective phases are controlled by

$$\frac{c(g,h)\,\text{satisfying Eq.71}}{c(g,h)\,\text{of the form Eq.72}} = H^2(G;U(1)) \tag{73}$$

So a $1d$ anomaly is classified by $H^2(G;U(1))$.

The Full-Story

We found above that a $1d$ theory has an $H^2(G;U(1))$ controlled anomaly, and before that an SPT phase was also classified by an element of $H^2(G;U(1))$. This hints at a relationship.

Consider coupling the $1d$ theory to the $2d$ theory as a boundary. The boundary only preserves G if the domain walls end topologically on the boundary. Now our theory has no unaccounted phase as we bring the operators together [12], see Figure 5.

The result is that: *the anomaly carried by the boundary theory is cancelled by anomalous gauge-variation of the bulk system.* This phenomena is known as *anomaly inflow.*

The full-story [31, 32], is as follows: For a connected Lie group G, 't Hooft anomalies in d dimensions are classified by $d+1$ dimensional topological actions. This is the canonical example of anomaly inflow because Chern-Simons actions are gauge invariant only up to a *Wess-Zumino-Witten term* living on the boundary. In the case that G is the opposite of connected (finite)

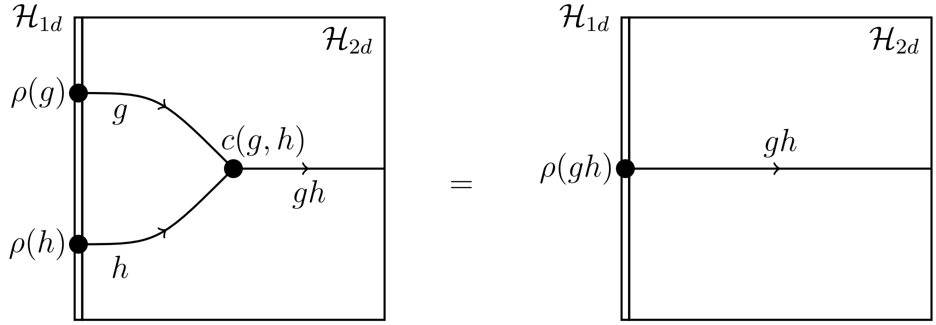


Figure 5: Anomaly inflow removes the ambiguous projective phase factor (anomaly) by using the phase attached to the trivalent junction. Recall that $\rho(gh) = c(g,h)\rho(g)\rho(h)$.

then we must make the *anomaly inflow assumption* [33], that states: *every anomaly in d dimensions can be cancelled by a $d+1$ dimensional inflow.* Topological actions in $d+1$ dimensions are known [28] to be classified by elements of $H^{d+2}(BG;\mathbb{Z})$, which is isomorphic to $H^{d+1}(BG;U(1))$ when G is finite. The classification of SPT phases relies on gauging G and integrating away everything except the G field, turning a $d+1$ dimension SPT phase into a topological theory. Hence, the boundary of an SPT phase either breaks G symmetry, or carries a field theory with the appropriate 't Hooft anomaly.

3.4 The Problem

Here we will outline the main problem we are trying to solve in this thesis and discuss some potential solutions.

Operations on a Theory

Thus far we have seen a number of operations that we can perform on a theory, and how they affect the partition function. The partition function $\mathcal{Z}_{g,h}$ is transformed under modular transformations (or the generators) as

$$S : \mathcal{Z}_{g,h} \mapsto \mathcal{Z}_{h,g} \tag{74}$$

$$T : \mathcal{Z}_{g,h} \mapsto \mathcal{Z}_{g,g+h}\,. \tag{75}$$

Similarly, adding an SPT phase maps

$$\mathcal{Z}_{g,h} \mapsto \epsilon(g,h)\mathcal{Z}_{g,h}\,. \tag{76}$$

Finally, we can gauge a symmetry, which is an involutive operation, which typically acts

$$\mathcal{Z}_{g,h} \mapsto \frac{1}{|G|} \sum_{g',h'} e^{if(g,h;g',h')} \mathcal{Z}_{g',h'} \, . \tag{77}$$

In some sense the modular transformations and adding an SPT phase are weak. The modular transformations don't change a complete theory at all, and two theories related by an SPT phase have the same dynamics and can't be distinguished at the level of the OPE. On the other hand, gauging is a strong operation, radically altering dynamics. Furthermore, if we gauge two theories that are related only by the addition of an SPT phase, they will produce dramatically different theories.

The Setup

Let's consider theories in $d = 2$, with $G = \mathbb{Z}_p \times \mathbb{Z}_p$ symmetry, where p is prime; we pick p prime so all non-trivial proper subgroups are isomorphic to $\mathbb{Z}_p$.

Under T and S transformations (hereby known as π_1 and π_2 respectively) the twisted partition functions transform as $\pi_1 : \mathcal{Z}_{\alpha,\alpha';\beta,\beta'} \mapsto \mathcal{Z}_{\alpha,\alpha+\alpha';\beta,\beta+\beta'}$ and $\pi_2 : \mathcal{Z}_{\alpha,\alpha';\beta,\beta'} \mapsto \mathcal{Z}_{\alpha',\alpha;\beta',\beta}$. Furthermore, we know that $H^2(G;U(1)) \cong \mathbb{Z}_p$, so that adding an SPT phase (denoted by applying operator S) acts by multiplying by a power of a p-th root of unity $S : \mathcal{Z}_{\alpha,\alpha';\beta,\beta'} \mapsto \omega^{\alpha\beta'-\beta\alpha'} \mathcal{Z}_{\alpha,\alpha';\beta,\beta'}$. Finally, gauging a symmetry acts as

$$g\, \mathcal{Z}_{\alpha,\alpha';\beta,\beta'} = \frac{1}{|G|} \sum_{\gamma,\delta} \omega^{\alpha\delta-\beta\gamma} \mathcal{Z}_{\gamma,\alpha';\delta,\beta'} \, . \tag{78}$$

Define $T_0 = \langle \pi_1, \pi_2, S \rangle$ to be the group of all weak transformations that can act on a theory, and define $T = \langle g, \pi_1, \pi_2, S \rangle$ to be the group of all transformations. The first questions we may set out to ask are

1. What are the groups T_0 and T?
2. Is there a geometrical interpretation of these operations?

Next let's consider the space of all twisted partition functions (background G-bundles) $\mathcal{C}$. We have a natural partitioning of $\mathcal{C}$ into equivalence classes under the orbits of T_0, these theories all have the same local dynamics. CFTs on orbifolds would be built from twisted partition functions in these equivalence classes, the classes being related by modular transformations and/or with some discrete torsion.

We may also be interested in how these equivalence classes interact under gauging. If we imagine $\mathcal{C}$ as being partitioned into patches, we could connect two patches of theory space with a bridge if two theories in that patch are related by gauging. That is, we could view equivalence classes as vertices in a graph, linked by an edge if they are related by a gauging (see Figure 6). This means the next question we could ask is,

3. What does the space of theories look like with $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry?

Our first attempts to answer this question are straightforward. We can treat a single twisted partition function $\mathcal{Z}_{\alpha,\alpha';\beta,\beta'}$, as a vector $|\alpha,\alpha';\beta,\beta'\rangle$ in a p^4 dimensional vector space over $\mathbb{Z}_p$. Then we can encode our operations as $p^4 \times p^4$-dimensional matrices and try to answer the questions. Right away,

$$S\,|\alpha,\alpha';\beta,\beta'\rangle = \omega^{\alpha\beta'-\alpha'\beta}\,|\alpha,\alpha';\beta,\beta'\rangle \tag{79}$$

$$\pi_1\,|\alpha,\alpha';\beta,\beta'\rangle = |\alpha,\alpha+\alpha';\beta,\beta+\beta'\rangle \tag{80}$$

$$\pi_2\,|\alpha,\alpha';\beta,\beta'\rangle = |\alpha',\alpha;\beta',\beta\rangle \tag{81}$$

so for all intents and purposes the phases out front do not "see" the kets. The operators don't commute, but since p is prime, it's not hard to show that through convoluted sequences of operations, that every $|\alpha,\alpha';\beta,\beta'\rangle$ can be appended with ω^k for $0 \le k < p$. This proves that

$$T_0 \cong \mathbb{Z}_p \times \mathrm{SL}_{\pm}(2,\mathbb{Z}_p)\,. \tag{82}$$

where $\pm$ denotes the fact that the matrices can have determinant ± 1.

Attempts to answer question 3 for $p = 2$ are straightforward in Mathematica and can be done analytically. The results are presented in Appendix A.3. With sufficient numerical programming trickery and a few hours of

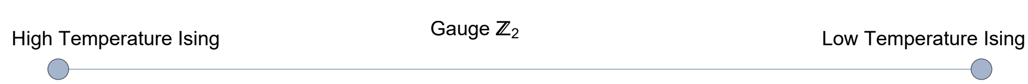


Figure 6: $\mathbb{Z}_2$ has no second group cohomology. The space of $\mathbb{Z}_2$ theories may be depicted with a two vertex graph, a $K_{1,1}$ graph.

calculation, the question can also be answered for $p = 3$. A more sophisticated approach clears the problem in general, as we will see, and with less computational difficulty.

4 TFTs and Boundary Conditions

In this final section, we will quickly discuss what a topological field theory is with the goal of understanding "where data is stored." We will also introduce the concept of an 't Hooft line as dual to a Wilson line, and see its relation to braiding. With this background, we will interpret our $2d$ theory from the previous section (or really any $2d$ theory) as boundary conditions for a $3d$ topological field theory, and use this intuition to solve the problem we posed at the beginning.

4.1 TFTs and Line Operators

Topological Field Theories

We will follow [28, 34] to first gain some insight into how TQFTs work. A *bordism* between two $n-1$ dimensional manifolds Σ and Σ' is an (oriented compact) n-dimensional manifold M which interpolates between Σ and Σ', i.e. the boundary is a disjoint union $\partial M = \Sigma \sqcup (-\Sigma')$ where the $-$ sign implies that Σ' has reversed orientation. By definition, Σ and Σ' are *cobordant* iff such a bordism M exists.

An n-dimensional oriented closed *topological quantum field theory* is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{F}} \,. \tag{83}$$

Here Bord_n is the category of $(n-1)$-dimensional manifolds whose morphisms are bordisms, and hence themselves n-manifolds, and $\mathrm{Vect}_{\mathbb{F}}$ are vector spaces over $\mathbb{F}$ with linear transformations. For our purposes $\mathbb{F} = \mathbb{C}$, and so each $(n-1)$-dimensional manifold Σ is associated with a Hilbert space of states $\mathcal{Z}(\Sigma) := \mathcal{H}_\Sigma$, and each n-manifold is associated with a linear transformation $\mathcal{Z}(M) : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$.

Since the concept of a bordism is so topological, we see such a mathematical definition certainly encapsulates the idea that a topological field theory should depend only on the topology, not the metric of the theory. The $\mathcal{Z}(M)$

tells us how fields evolve given their configuration at the boundary Σ. i.e. it tells us that a field Φ on M which looks like φ on Σ can be written as

$$\mathcal{Z}(M)(\varphi) = \int_{\Phi|_\Sigma=\varphi} D\Phi\, e^{-S[\Phi]}\,. \tag{84}$$

That is, setting boundary conditions for fields is the same as picking a state.

Wilson and 't Hooft Lines

Given a G-gauge theory with connection A on a principal G-bundle P over M, a closed (oriented) curve γ with basepoint $x \in M$ defines parallel transport in P [23, 35], i.e. it gives an automorphism of P_x or holonomy around the curve. Let R be a rep of G, then for any G-gauge theory we always have the *Wilson loop* operator

$$W_{\gamma,R}(A) = \mathrm{Tr}_R(\mathrm{Hol}_\gamma(A)) = \mathrm{Tr}_R\, P\exp\left(i\int_\gamma A\right)\,. \tag{85}$$

For an open curve γ with endpoints p and q, we can think of the *Wilson line* as the matrix parallel transporting representation-R data from P_p to representation-R data in P_q (note that the fibers have a G action, and hence naturally decompose into a sum of reps of G). These are sometimes called *order operators* in statistical mechanics [36].

Similarly, we also have a notion of a *disorder operator* or *'t Hooft line* [37] $T_{\gamma,\mu}$ along γ, which modifies the connection to have a simple pole around γ. We do this by effectively replacing every A-field with a B-field satisfying $dA \sim \star dB$ as explained in [36] and Vol. 2 Lecture 10 of [38]. The key for our purposes is that 't Hooft lines are labelled by cocharacters $\mu : U(1) \to G$.

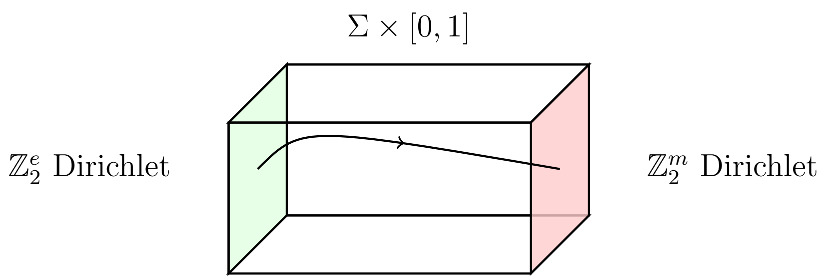
Furthermore, noting that Wilson lines are labelled by characters in the case G is abelian, we find a sort of braiding relation between the Wilson lines [37] given based on if the lines have non-trivial linking configuration ℓ

$$W_{\gamma,\chi}T_{\gamma',\mu} = \langle\mu,\chi\rangle^{\ell(\gamma,\gamma')}T_{\gamma',\mu}W_{\gamma,\chi}\,. \tag{86}$$

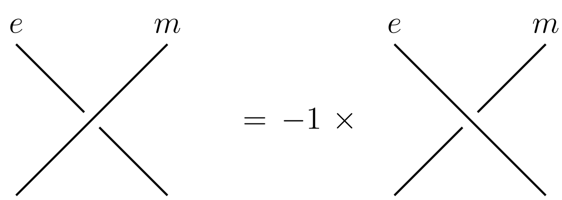
The fact that Wilson lines and 't Hooft lines look to be interchangeable is a manifestation of an *electric-magnetic duality*. Dualities of this flavour arise in other contexts, like *Montonen-Olive duality* [39], and can be understood better through the geometric Langlands program [36].

4.2 2d Theories on the Boundary

We are going to try to interpret our study of $2d$ theories as being boundary conditions for a $3d$ theory. The goal is to give geometrical/topological insight into the operations we were performing in the $2d$ case, a common strategy providing deep insights into $2d$ field theories [40, 41, 42]. Our discussion will follow [12] and [43].



Consider a $3d$ $\mathbb{Z}_2$ gauge theory[7] on a slab $M = \Sigma \times [0,1]$. This theory has two kinds of line operators, Wilson lines and 't Hooft lines, labelled by e and m respectively. Conveniently, in this case, there is only one non-trivial line of each type since there is only one non-trivial rep and group element of $\mathbb{Z}_2$ respectively. Their braiding relations look like



We can place Dirichlet $\mathbb{Z}_2^e$ boundary conditions on one side of the slab Σ, and Dirichlet $\mathbb{Z}_2^m$ boundary conditions (aka. Neumann $\mathbb{Z}_2^e$ boundary conditions) on $-\Sigma$. Now suppose we have our $2d$ theory T with, say, $G = \mathbb{Z}_2^e$ symmetry. Then we can put this theory on the Σ boundary of our slab. If we imagine gauging the $\mathbb{Z}_2^e$ symmetry of the setup, then it will disappear, and we claim that all that is left is a $\hat{G} = \mathbb{Z}_2^m$-symmetric theory, i.e. $T//\mathbb{Z}_2^e$.

We can see this at the level of the partition function. The $3d$ topological gauge theory on M associates $\mathcal{H}_\Sigma^{3d}$ to the surface Σ. There are various line

[7]The best example of this is the toric code which we outline in Appendix A.2, in particular, we will focus on discussions with "gapped boundary conditions."

operators that can act on this space; for example, a cycle $a \in H_1(\Sigma, \mathbb{Z}_2)$ can be labelled with a Wilson line operator $L_e(a)$, and similarly for $L_m(a)$. If Σ is equipped with Dirichlet $\mathbb{Z}_2^e$ boundary conditions given by $H_1(\Sigma, \mathbb{Z}_2)$, this creates a state for the $3d$ theory specified by the background value v, $|e, v\rangle$, and similarly for $|m, w\rangle$ (the basis diagonalizing the 't Hooft magnetic line operators). Where

$$L_e(a)\,|e, v\rangle = (-1)^{\int a\wedge v}\,|e, v\rangle \tag{87}$$

$$L_m(b)\,|m, w\rangle = (-1)^{\int b\wedge w}\,|m, w\rangle\,. \tag{88}$$

Note we use $\wedge$ by first taking the Poincaré dual. Using our commutation relation

$$L_e(a)L_m(b) = (-1)^{\int a\wedge b}L_m(b)L_e(a)\,, \tag{89}$$

we arrive at

$$\langle e, v|m, w\rangle = (-1)^{\int v\wedge w}\,. \tag{90}$$

So now, the partition function can be read from left to right on our slab

$$\mathcal{Z}_{T//\mathbb{Z}_2^e}[\Sigma, w] = \sum_v \mathcal{Z}_T[\Sigma, v]\,\langle e, v|m, w\rangle \tag{91}$$

where the $\sum_v$ comes from summing over all background $\mathbb{Z}_2$ configurations when we gauge the bulk [12]. We recognize this as our result from Equation 35. If instead we would have used (electric) Dirichlet boundary conditions on the opposite side, the partition function would be

$$\mathcal{Z}_T[\Sigma, w] = \sum_v \mathcal{Z}_T[\Sigma, v]\,\langle e, v|e, w\rangle \tag{92}$$

In summary, if we use a $2d$ theory as a boundary condition for a $3d$ theory, gauging leads to a new theory based on the other boundary conditions. i.e.

$${}_T[0,1]_{\mathrm{Dir}} = T \tag{93}$$

$${}_T[0,1]_{\mathrm{Neu}} = T//G \tag{94}$$

We also see that Dirichlet boundary conditions for the electric lines means that the bundles are trivial, so that these are boundary conditions with freedom for $\hat{G} \times \{0\}$ line operators to end. Conversely, Neumann boundary conditions correspond to $\{0\} \times G$.

4.3 The Solution

With this discussion in mind, we will now say words that look familiar but are far beyond the discussion of the author to explain:

> Symmetries of three-dimensional topological field theories are naturally defined in terms of invertible topological surface defects. Symmetry groups are thus Brauer–Picard groups [5].

Furthermore:

> Let A be a finite abelian group. Then [it is shown] that
>
> $$\mathrm{BrPic}(\mathrm{Vec}_A) \cong O(A \oplus \hat{A}, q) \tag{95}$$
>
> where $O(A \oplus \hat{A}, q)$ is the group of automorphisms of $A \oplus \hat{A}$ preserving the canonical quadratic form [2].

What these statements are saying is that the group $T = \langle g, S, \pi_1, \pi_2 \rangle$ describing a theory with abelian symmetry group A is precisely $O(A \oplus \hat{A}, q)$, where q is the natural pairing between characters and co-characters.

Take $A = \mathbb{Z}_p^k$ for example, then $\hat{A} \cong \mathbb{Z}_p^k$, and the quadratic form $q : A \times \hat{A} \to U(1)$ is easily shown to be given by

$$q(a, \hat{a}) = \exp(2\pi i a \cdot \hat{a}/k) \tag{96}$$

where $a = (a_1, \ldots, a_k) \in A$ and $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_k) \in \hat{A}$. Our automorphisms of $A \times \hat{A}$ are thus the automorphisms of $\mathbb{Z}_p^{2k}$ fixing the regular dot product. By definition this is just

$$T \cong O(k, k; \mathbb{Z}_p)\,. \tag{97}$$

This approach also gives us more insight into the appropriate basis for calculations. Indeed, the problem with trying to answer question 3 before was that p^4 dimensional matrices get quite hard to generate due to their size, and numerical precision becomes an issue as we deal with prime roots of unity as entries. If $|A, B; C, D\rangle$ was our old basis/set of partition functions, the new natural basis to consider is

$$|(A, B), (C, D)\rangle_\star = \sum_{X,Y \in \mathbb{Z}_p} q((X, Y), (C, D))\, |A, B; X, Y\rangle\,. \tag{98}$$

It is straightforward to show that in this basis

$$g\,|(A,B),(C,D)\rangle_\star = |(C,B),(A,D)\rangle_\star \tag{99}$$

$$S\,|(A,B),(C,D)\rangle_\star = |(B,A),(D,C)\rangle_\star \tag{100}$$

$$\pi_1\,|(A,B),(C,D)\rangle_\star = |(A,A+B),(C-D,D)\rangle_\star \tag{101}$$

$$\pi_2\,|(A,B),(C,D)\rangle_\star = |(A,B),(C-B,D+A)\rangle_\star\,. \tag{102}$$

Which gives us a computationally tractable basis to work with. In particular, for $G = \mathbb{Z}_p \times \mathbb{Z}_p$ all of our matrices implementing the operations are 4×4 matrices over $\mathbb{Z}_p$. The results are catalogued in Appendix A.3.

In summary, we find for $G = \mathbb{Z}_p \times \mathbb{Z}_p$ that

1. $T_0 \cong \mathbb{Z}_p \times SL(2,\mathbb{Z}_p)$, $T \cong O(2,2;\mathbb{Z}_p)$.
2. There is a geometric interpretation of the operations in terms of actions in a $3d$ topological field theory.
3. The space of theories form a bipartite graph, formed as the orbits of the T_0 cosets of T under g.

5 Conclusion

In this essay we tried to form a pedagogical bridge from the end of a first course in CFT to modern high energy physics and condensed matter literature. Our driving example was to understand the space of $2d$ theories with $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry at the end. We began by covering modular invariance, orbifolds, and gauging in an abstract way which has numerous applications in the CFT and string literature [14, 27]. After this, we moved on to discuss anomalies and their generalities, and the seemingly unrelated technique of representing G-bundles with flat connections as networks of defects [15]. We then tied the subjects together by connecting to modern research on SPT phases [3, 29, 31, 32, 33]. After this we discussed the setup for our driving problem, and tried to understand how the collection of orbifolds of theories with $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry is arranged.

To solve this problem we again had to return to ideas prevalent in modern research, we saw forms of the electric-magnetic duality, and how we can couple theories in two dimensions to three dimensional topological theories. Using these techniques we solved our driving problem by interpreting our 4 operations in terms of a much simpler set of partition functions, more naturally suited to the three-dimensional setup: the space of $2d$ theories with $\mathbb{Z}_p \times \mathbb{Z}_p$ form bipartite graphs, and the weak and strong group operations are isomorphic to $\mathbb{Z}_p \times \mathrm{SL}_\pm(2, \mathbb{Z}_p)$ and $O(2, 2; \mathbb{Z}_p)$ respectively.

Future work will be to extend this formalism to theories with fermions, which are very deeply connected to the $\mathbb{Z}_2$ case we studied already, except with spin structures and their associated technicalities arising along the way.

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A Calculations

A.1 Compactified Boson

We will quickly cover the case of a boson compactified on a circle following very closely to [9]. We will be brief because most of this we have seen before in the case of PDEs, Fourier analysis, string theory, and so on.

Begin by considering a free boson in the plane that takes values on $\mathbb{S}^1$

$$X(z,\bar{z}) \sim X(z,\bar{z}) + 2\pi R n, \qquad n \in \mathbb{Z} \tag{103}$$

where n is the *winding number* around the circle. We expand as

$$X(z,\bar{z}) = x_0 - i(j_0 \ln z + \bar{j}_0 \ln \bar{z}) + i \sum_{n\neq 0} \frac{1}{n} \left(j_n z^{-n} + \bar{j}_n \bar{z}^{-n} \right) . \tag{104}$$

Our field can have (what would typically have been non-trivial monodromy) that is now trivial because we are on the circle

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = X(z,\bar{z}) + 2\pi Rn\,, \tag{105}$$

which implies

$$j_0 - \bar{j}_0 = Rn\,, \tag{106}$$

where $n = 0$ if we weren't on a circle. This tells us that energy ground states of our boson can be non-trivially charged. In particular, we choose a basis $\{|\Delta, n\rangle\}$ satisfying

$$j_0\,|\Delta,n\rangle = \Delta\,|\Delta,n\rangle\,, \qquad \bar{j}_0\,|\Delta,n\rangle = (\Delta - Rn)\,|\Delta,n\rangle \tag{107}$$

on our ground states, where Δ is a continuous charge variable (for now). Noticing that the action of all other modes is exactly the same, we can use our result about the free-boson without the center-of-mass term, and then throw on our new zero-mode partition function contribution

$$\mathcal{Z}_{\text{circ.}}(\tau,\bar{\tau}) = \mathcal{Z}'_{\text{bos.}}(\tau,\bar{\tau}) \sum_{\Delta,n} \left\langle \Delta, n \middle| q^{\frac{1}{2}j_0^2}\bar{q}^{\frac{1}{2}\bar{j}_0^2} \middle| \Delta, n \right\rangle \tag{108}$$

$$= \frac{1}{|\eta(\tau)|^2} \sum_{\Delta,n} q^{\frac{1}{2}\Delta^2}\bar{q}^{\frac{1}{2}(\Delta - Rn)^2} \tag{109}$$

Demanding T-invariance we see that Δ is actually a discrete variable satisfying $\Delta = m/R + Rn/2$ for $m \in \mathbb{Z}$. This gives the final result that

$$j_0\,|m,n\rangle = \left(\frac{m}{R} + \frac{Rn}{2}\right)|m,n\rangle\,, \qquad \bar{j}_0\,|m,n\rangle = \left(\frac{m}{R} - \frac{Rn}{2}\right)|m,n\rangle \tag{110}$$

and

$$\mathcal{Z}_{\text{circ.}}(\tau,\bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n} q^{\frac{1}{2}\left(\frac{m}{R}+\frac{Rn}{2}\right)^2}\bar{q}^{\frac{1}{2}\left(\frac{m}{R}-\frac{Rn}{2}\right)^2} \tag{111}$$

where $n \neq 0$ states corresponding to the states winding around the circle n-times, and $m \neq 0$ states corresponding to momentum states with center-of-mass momentum m/R. S-invariance follows from applying the Poisson resummation formula twice.

A.2 The Toric Code

The following is an example of a $3d$ topological $\mathbb{Z}_2$ gauge theory, which has a $2d$ theory at its boundary. It is the $2d$ quantum Ising model (dual to the $3d$ classical ising model). We will see what happens when we try to gauge its global $\mathbb{Z}_2$ symmetry, following the original work of [44] and extended reviews of [45, 46].

The Setup and Missing Operators

Our setup is as follows: we start with an $L \times L$ square lattice with periodic boundary conditions (L^2 independent faces and vertices and $2L^2$ edges). We place a 2-level quantum system on each edge/link $\ell = (\mathbf{n}, \hat{\mu})$, which we formally label by a site $\mathbf{n}$ it connects to and unit-vector it extends along. We will write $\ell \in \mathbf{n}$. This places a Hilbert-space $\mathbb{C}^2$ on each link, or in total

$$\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes 2L^2} . \tag{112}$$

The Hamiltonian of the quantum $\mathbb{Z}_2$ gauge theory is

$$H = -J \sum_{\mathbf{p}} B_{\mathbf{p}} - g \sum_{\ell} \sigma_\ell^x , \tag{113}$$

where the *plaquette operator* returns the product of spins around a square $\mathbf{p}$, otherwise known as the $\mathbb{Z}_2$ flux through the plaquette

$$B_{\mathbf{p}} = \prod_{\ell \in \mathbf{p}} \sigma_\ell^z . \tag{114}$$

Next we declare a gauge structure by defining an operator on sites,

$$A_{\mathbf{n}} = \prod_{\ell \in \mathbf{n}} \sigma_l^x , \tag{115}$$

which is a gauge-version of our global $\mathbb{Z}_2$ symmetry, because it flips all qubits linked only to site $\mathbf{n}$.

We can construct the space of all gauge-invariant states by projecting with

$$\mathcal{P} = \prod_{\mathbf{n}} \frac{1 + A_{\mathbf{n}}}{2} = \prod_{\mathbf{n}} \mathcal{P}_{\mathbf{n}} \tag{116}$$

to find

$$\mathcal{H}_{\text{gauge}} = \mathcal{P}\mathcal{H}\,. \tag{117}$$

$\mathcal{P}_{\mathbf{n}}$ acts by selecting the $+1$ eigenspaces of the $A_{\mathbf{n}}$ operator, so $\dim \mathcal{P}_{\mathbf{n}}\mathcal{H} = \frac{1}{2}\dim\mathcal{H}$, and in particular

$$\dim \mathcal{H}_{\text{gauge}} = \frac{1}{2^{L^2-1}}\dim\mathcal{H} = 2^{L^2+1}\,, \tag{118}$$

where we note that there are only L^2-1 independent gauge transformations because $\prod_{\mathbf{n}} A_{\mathbf{n}} = 1$.

We see since $A_{\mathbf{n}}$ and $B_{\mathbf{p}}$ can only share an even number of bonds that $[A_{\mathbf{n}}, B_{\mathbf{p}}] = 0$, and that different plaquette operators commute. So we can label states by $B_{\mathbf{p}}$ quantum numbers for $\mathcal{H}_{\text{gauge}}$. We note similar to before that $\prod_{\mathbf{p}} B_{\mathbf{p}} = 1$, so that there are L^2-1 indepedent plaquettes. We see if we want to span the whole gauge-invariant Hilbert space, we are short at least two operators.

Phase Transitions and Topological Order

It was originally shown [44] that this model exhibits a phase transition with the help of the Wilson loop[8]

$$W_{\mathcal{C}} = \prod_{\ell\in\mathcal{C}} \sigma_\ell^z \tag{119}$$

around a closed contour $\mathcal{C}$ in the lattice. In the limits:

1. $g/J \to \infty$. *Confining phase*, $W_{\mathcal{C}}$ correlation functions decay with an area law $\sim \exp(-\alpha A_{\mathcal{C}})$, which can be seen as the paramagnetic state. Large decay is attributed to large fluctuations of $\mathbb{Z}_2$ flux.
2. $g/J \to 0$. *Deconfined phase*, $W_{\mathcal{C}}$ follows a perimeter law due to expulsion of $\mathbb{Z}_2$ flux in the plaquettes, just as in the Meissner effect for superconductors, represented by all $B_{\mathbf{p}}$ having $+1$ eigenvalue.

This analysis is not generalizable because if we add coupling to matter fields all decay laws will become perimeter laws, but certainly in this case it shows a phase transition occurs. Historically this was unique, because it cannot arise from a spontaneous symmetry breaking as there was no *local* order

[8] The paper by Wegner was around 3 years prior to the introduction of the Wilson loop in gauge theories, so we should call it a Wegner loop.

parameter characterizing the phase transition; and we know from Elitzur's theorem there can't be. Could this be linked to our missing operators?

On the plane, we wouldn't be missing any operators, the missing operators came from the fact that our square mesh was on a torus. So we can assume that somehow the geometry/topology of the torus is to blame. We also can see by putting two plaquette operators together, next to one another, that they form into an effectively bigger plaquette with area 2, but they are all homotopically equivalent to a point, and no configuration or number of plaquettes can change this; this is the giveaway. Define the two non-contractible loops of operators

$$W^z_{x,y} = \prod_{\ell \in \mathcal{C}_{x,y}} \sigma^z_\ell \tag{120}$$

where x means that the loop goes around (something homotopic to) a horizontal cycle of the torus, and y is analogous for a vertical cycle. We can also define the analogous $W^x_{x,y}$, and find that these loops satisfy the Pauli algebra, so indeed they are the remaining missing spin-variables.

Now define 't Hooft line operators, dual to the Wilson line operators, as

$$V^x_{x,y} = \prod_{\ell \perp \bar{\mathcal{C}}_{x,y}} \sigma^x_\ell \,, \tag{121}$$

where now the cycles are horizontal or vertical, but defined on the dual lattice, so that the σ^x_ℓ applies to the links crossed by the 't Hooft lines. We can verify that $[V^x_x, H] = \left[V^x_y, H\right] = 0$, but more topologically important are the crossing relations for the lines

$$W^z_x V^x_y = -V^x_y W^z_x, \qquad W^z_y V^x_x = -V^x_x W^z_y \,, \tag{122}$$

which basically say if a Wilson line crosses an 't Hooft line, it's the same as an 't Hooft line crossing a Wilson line up to a $\mathbb{Z}_2$ phase factor (for $\mathbb{Z}_n$ it could be an n-th root of unity). See Figure 7.

In the deconfined phase we investigate the ground state $|0\rangle = \mathcal{P}\,|\Uparrow\rangle$, where $|\Uparrow\rangle$ is the state with all spins up. In particular, we note $|0\rangle$ is an eigenstate of $W^z_{x,y}$ with eigenvalues $+1$. Now, if we lay down 't Hooft lines first, our

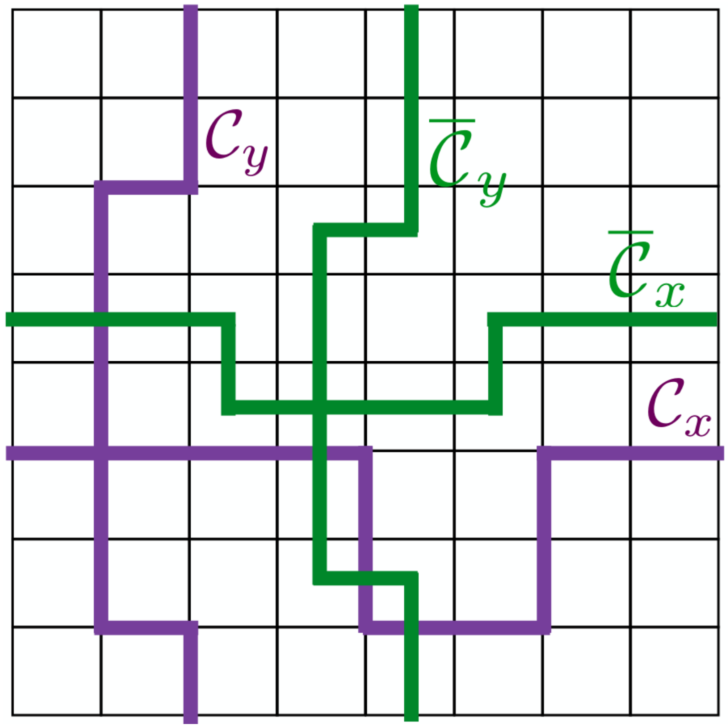


Figure 7: In purple we see Wilson-lines around the different cycles of the torus. In green we see 't Hooft lines on the dual lattice. Image taken from [46].

crossing relations (for closed loops) tell us that

$$|0\rangle \sim W_x^z = 1, \quad W_y^z = 1, \tag{123}$$

$$V_x^x |0\rangle \sim W_x^z = 1, \quad W_y^z = -1, \tag{124}$$

$$V_y^x |0\rangle \sim W_x^z = -1, W_y^z = 1, \tag{125}$$

$$V_x^x V_y^x |0\rangle \sim W_x^z = -1, W_y^z = -1, \tag{126}$$

are all states with the same energy, but differing $\mathbb{Z}_2$ flux through the torus cycles. The ground state is 4-fold degenerate, and the states of the system are built on top of the 4 ground states, which cannot be related except by these non-local operators.

The modern view is that the deconfined phase has $\mathbb{Z}_2$ *topological order*. Phases with topological order are characterized as having stable low-lying excitations which cannot be created by any local operator, or *superselection sectors*. Furthermore, the collection of gauge-transformations do not constitute all operators that commute with the Hamiltonian, and their eigenspaces do not span the Hilbert space; the full set of operators will have topology-dependent operators. The ground state is characterized by its robustness, i.e. topological order is protected against any perturbations[9].

Kramers-Wannier "Duality"

Let's restrict our attention to the $W^z_x = W^z_y = 1$ sector. An operator on a site of the dual lattice is just an operator on a face, so we label it with a $\mathbf{p}$.

We begin by defining $\mu_1(\mathbf{p}) = B_{\mathbf{p}}$. Next, we define an operator

$$\mu_3(\mathbf{p}, \hat{v}) = \prod_{\ell}^{\mathbf{p}} \sigma^x_\ell \tag{127}$$

where $\hat{v} = \hat{x}$ means the product is a semi-infinite line on the plane from $\mathbf{p}$ out to infinity, and similarly for $\hat{y}$. We can do the same on the torus by just fixing a point for all of these lines to terminate. This operator acts to flip (or disorder) all the spins outward from $\mathbf{p}$, creating a string of domain walls.

The key notes are that for $\ell \in \mathbf{p}$

$$\sigma^x_l = \mu_3(\mathbf{p}, \hat{v})\mu_3(\mathbf{p} - \hat{v}, \hat{v}) \tag{128}$$

and that $\mu_1(\mathbf{p})$ and $\mu_3(\mathbf{p}, \hat{v})$ satisfy the Pauli algebra, and commute if $\mathbf{p} \neq \mathbf{p}'$, so they are simply spin variables on the dual lattice.

The result is that under this map

$$H_{\mathbb{Z}_2} \mapsto -g \sum_{\mathbf{p}, \hat{v} \in \{\hat{x}, \hat{y}\}} \mu_3(\mathbf{p}, \hat{v})\mu_3(\mathbf{p} - \hat{v}, \hat{v}) - J \sum_{\mathbf{p}} \mu_1(\mathbf{p}) \tag{129}$$

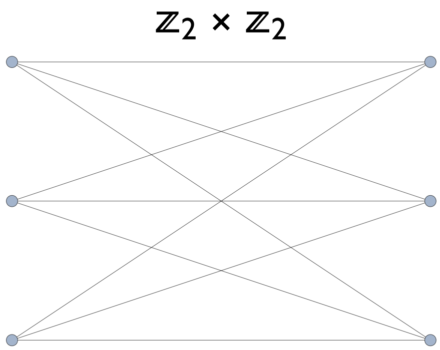
so that: *the high temperature Ising gauge theory is dual to the low temperature Ising model, and the low temperature Ising gauge theory is dual to the high temperature Ising model.*

This manifests itself in the $2d$ theory as well. A spatial boundary of the $2 + 1d$ toric code we established above is the classical statistical mechanics $2d$ Ising model. Meanwhile, a spacetime boundary is the $1 + 1d$ spin-chain representation of the Ising model [47].

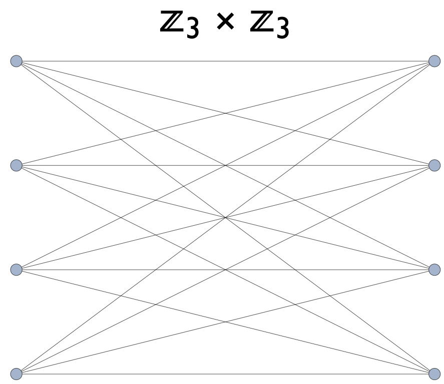
[9] As opposed to topological insulators which are only robust for time-reversal and $U(1)$ perturbations.

A.3 Results for $\mathbb{Z}_p \times \mathbb{Z}_p$ Theories

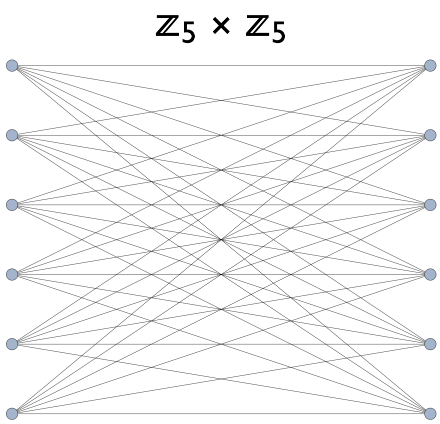
The result is that $G = \mathbb{Z}_p \times \mathbb{Z}_p$ produces a bipartite graph. We exhibit the first 4 results below:



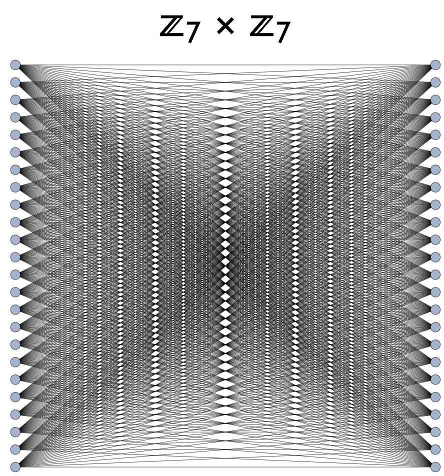
(a) $|T_0| = 12$, $|T| = 72$. 6 Distinct theories.



(b) $|T_0| = 144$, $|T| = 1152$. 8 Distinct theories.



(c) $|T_0| = 1200$, $|T| = 14,400$. 12 Distinct theories.



(d) $|T_0| = 4704$, $|T| = 225,792$. 48 Distinct theories.

B Special Functions and Mathematical Facts

B.1 Special Functions and Modular Properties

The Dedekind η-Function

For any τ in the upper-half complex plane, we define the *nome* to be $e^{2\pi i\tau}$. The *Dedekind η-function* is a function on the upper-half complex plane, defined in-terms of the *nome* it reads

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{130}$$

it satisifies

$$T : \eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau) \tag{131}$$

$$S : \eta\left(-\tfrac{1}{\tau}\right) = \sqrt{\tfrac{\tau}{i}}\eta(\tau) \tag{132}$$

The θ-Function

The theta-function is a special function with applications to number theory and elliptic equations.

$$\vartheta(z;\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} e^{2\pi i n z} \tag{133}$$

From this we define

$$\vartheta_2(\tau) := q^{\frac{1}{8}} \vartheta\left(\tfrac{1}{2}\tau;\tau\right) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \tag{134}$$

$$\vartheta_3(\tau) := \vartheta(0;\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} \tag{135}$$

$$\vartheta_4(\tau) := i q^{\frac{1}{8}} \vartheta\left(\tfrac{1}{2}\tau + \tfrac{1}{2};\tau\right) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} \tag{136}$$

which have product representations more useful for us

$$\vartheta_2(\tau) = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2 \tag{137}$$

$$\vartheta_3(\tau) = \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-\frac{1}{2}})^2 \tag{138}$$

$$\vartheta_4(\tau) = \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-\frac{1}{2}})^2 \tag{139}$$

satisfying

$$T : \vartheta_2(\tau+1) \mapsto e^{\frac{\pi i}{4}} \vartheta_2(\tau) \tag{140}$$

$$T : \vartheta_3(\tau+1) \mapsto \vartheta_4(\tau) \tag{141}$$

$$T : \vartheta_4(\tau+1) \mapsto \vartheta_3(\tau) \tag{142}$$

$$S : \vartheta_2\left(-\tfrac{1}{\tau}\right) \quad \mapsto \sqrt{\tfrac{\tau}{i}} \vartheta_4(\tau) \tag{143}$$

$$S : \vartheta_3\left(-\tfrac{1}{\tau}\right) \quad \mapsto \sqrt{\tfrac{\tau}{i}} \vartheta_3(\tau) \tag{144}$$

$$S : \vartheta_4\left(-\tfrac{1}{\tau}\right) \quad \mapsto \sqrt{\tfrac{\tau}{i}} \vartheta_2(\tau) \tag{145}$$

and finally

$$\frac{1}{2} \vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = \eta(\tau)^3 \,. \tag{146}$$

B.2 Group Cohomology

The definition of group cohomology we may be most familiar with from physics is the presentation most closely resembling the presentation of de Rham cohomology in terms of cochains [12, 48].

Let M be a G-module, the (abelian group of inhomogeneous) *n-cochains* with values in M are

$$C^n(G; M) = \{\text{functions } f : G^n \to M\} \,. \tag{147}$$

We define the *coboundary operator* $d^n : C^n(G;M) \to C^{n+1}(G;M)$ by

$$\begin{aligned}(d^n f)(g_1,\dots,g_{n+1}) &= g_1 \cdot f(g_2,\dots,g_n) \\ &+ \sum_{j=1}^{n}(-1)^j f(g_1,\dots,g_{j-1},g_j g_{j+1},\dots,g_{n+1}) \\ &+ (-1)^{n+1} f(g_1,\dots,g_n)\,,\end{aligned} \tag{148}$$

which we assure satisfies $d^2 = 0$ (or 1 multiplicatively). As per usual, we define the group of *n-cocycles*

$$Z^n(G;M) = \mathrm{Ker}(d^n) \tag{149}$$

and *n-coboundaries*

$$B^n(G;M) = \mathrm{Im}(d^{n-1})\,. \tag{150}$$

So that finally, the *n-th cohomology group* is

$$H^n(G;M) = \frac{Z^n(G;M)}{B^n(G;M)}\,. \tag{151}$$

As an example, we will often study $H^2(G;U(1))$, and all of our G-actions are trivial. The closed forms $c(g,h)$ must satisfy

$$1 = (dc)(g,h,k) = c(h,k) \times c(gh,k)^{-1} \times c(g,hk) \times c(g,h)^{-1}\,. \tag{152}$$

Meanwhile, the exact forms are those $(db)(g,h)$ of the form

$$(db)(g,h) = b(h) \times b(gh)^{-1} \times b(g)\,. \tag{153}$$

It is a well-known fact that $H^2(\mathbb{Z}_n \times \mathbb{Z}_m; U(1)) \cong \mathbb{Z}_{\gcd(n,m)}$, so that, in particular $H^2(\mathbb{Z}_k \times \mathbb{Z}_k; U(1)) \cong \mathbb{Z}_k$. So for example, our discrete torsion/SPT phases on the twisted sector $\mathcal{Z}_{(g,h),(g',h')}$ of a $\mathbb{Z}_k \times \mathbb{Z}_k$ orbifold are $\zeta^{m(gh'-hg')}$, where ζ is a k-th root of unity [27].