# WZW and Minimal Models

## for Reading Group Meeting

Justin Kulp

Perimeter Institute for
Theoretical Physics

03/Apr/2020

# Overview

# Minimal Models

# Minimal Models: Verma Modules and Unitarity

- Denote by $V(c,h)$ the Verma module generated by $\{L_n\}$ for some value $c$ of central charge and highest weight $h$.
- Pick basis states $|i\rangle$ of $V(c,h)$ at some fixed level $l$, then consider $M_{ij}^{(l)} = \langle i|j\rangle$. Kac shows

$$\det M^{(l)} = \alpha_l \prod_{\substack{r,s\geq 1 \\ rs \leq l}} (h - h_{r,s}(c))^{p(l-rs)} \tag{1}$$

  - If we tune $h = h_{r,s}(c)$, then there is a null state and $V(c,h)$ is reducible, with its first null state at level $rs$.

$$h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 \tag{2}$$

$$h_0 = \frac{1}{24}(c-1) \tag{3}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \tag{4}$$

The state

$$|\chi\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|h\rangle \tag{5}$$

is null iff

$$h = \frac{1}{16}\left\{5 - c \pm \sqrt{(c-1)(c-25)}\right\}. \tag{6}$$

In terms of fields, this lets us write

$$\chi(z) = \phi^{(-2)}(z) - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\phi(z). \tag{7}$$

Correlation functions must satisfy the induced differential equation

$$\left\{\sum_{i=1}^{N}\left[\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{h_i}{(z-z_i)^2}\right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right\}\langle\phi(z)X\rangle = 0 \tag{8}$$

In particular three-point function must satisfy it, restricting the operator algebra.

- In general, if $h = h_{r,s}$ then there is a null vector at level $rs$ and we (BPZ) may work to get the fusion rules

$$\phi_{(r_1,s_1)} \times \phi_{(r_2,s_2)} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 \pmod 2}}^{k=r_1+r_2-1} \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=1 \pmod 2}}^{l=s_1+s_2-1} \phi_{(k,l)} \tag{9}$$

  - Degenerate conformal families (conformal family modulo the null vector at level $rs$) form a closed operator algebra.
  - Can still be an infinite number of families in the theory.
- Can we make it so that there are only a finite number of conformal families?

- Yes! Given positive co-prime integers $p > p' \geq 2$, one gets a finite number of local fields with well-defined scaling behaviour if

$$c(p,p') = 1 - 6\frac{(p-p')^2}{pp'} \tag{10}$$

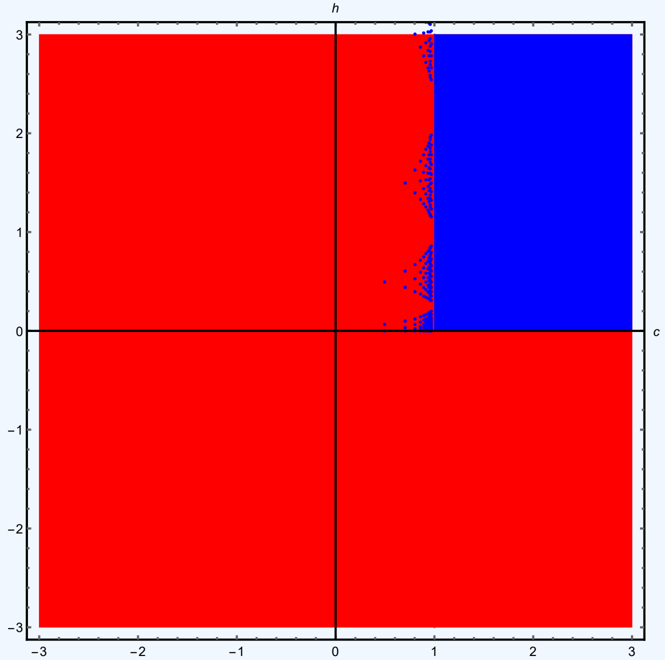
  - Unitary if $(p,p') = (m+1, m)$.

- This means that any highest weights would take values in

$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'} \,. \tag{11}$$

  - This has the property that $h_{r,s} = h_{p'-r,p-s}$
  - Finite set of weights $h_{r,s}$ is often delimited by $1 \leq r < p'$ and $1 \leq s < p$, but half are redudant due to index relation
  - A closed operator algebra of $(p-1)(p'-1)/2$ distinct fields.

- Such theories are called *minimal models*.


h
3
2
1
0
−1
−2
−3
c
−3
−2
−1
0
1
2
3

# WZW Models

# WZW Models: General Facts

- Set of holomorphic currents $J^a(z)$ subject to the OPE

$$J^a(z)\,J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{i{f^{ab}}_c}{z-w}\,J^c(w) \tag{12}$$

  - ${f^{ab}}_c$ define a Lie-algebra $\mathfrak{g}$
  - If $\mathfrak{g}$ is simple, $\kappa^{ab} = k\delta^{ab}$ for some $k$
  - Laurent modes have structure of *affine Kac-Moody algebra*

$$[J^a_m, J^b_n] = km\delta^{ab}\delta_{m+n,0} + i{f^{ab}}_c J^c_{m+n} \tag{13}$$

- *Sugawara construction* gives $T(z)$ and tells us

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee} \tag{14}$$

  - $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$

# WZW MODELS: $\mathfrak{su}(2)_k$

- Consider $\mathfrak{su}(2)_k$

$$[J^3_m, J^3_n] = \frac{k}{2} m \delta_{m+n,0} \,, \tag{15}$$
$$[J^3_m, J^\pm_n] = \pm J^\pm_{m+n} \,, \tag{16}$$
$$[J^+_m, J^-_n] = km\delta_{m+n,0} + 2J^3_{m+n} \,. \tag{17}$$

- HWS $|j\rangle$ corresponding to integrable rep satisfies

$$J^a_n |j\rangle = 0 \quad \text{for} \quad n > 0 \,, \tag{18}$$
$$J^+_0 |j\rangle = 0 \,, \tag{19}$$
$$J^3_0 |j\rangle = j |j\rangle \,. \tag{20}$$

Here $0 \leq j \leq k/2$

- Verma-module from lowering operators is not irreducible.

$$|\mathcal{N}\rangle = (J^+_{-1})^{k+1-2j} |j\rangle = 0 \,. \tag{21}$$

- Coset construction takes a WZW-model $\mathfrak{g}_k$ and a sub-algebra $\mathfrak{h}_{k'} \subset \mathfrak{g}_k$ to be gauged.
- Chiral algebra consists of all fields in $\mathfrak{g}_k$ model, which have regular OPE with *Kac-Moody fields* of $\mathfrak{h}_{k'}$.
- $T^{\mathfrak{g}/\mathfrak{h}} = T^{\mathfrak{g}} - T^{\mathfrak{h}}$ and $c_{\mathfrak{g}/\mathfrak{h}} = c_{\mathfrak{g}} - c_{\mathfrak{h}}$.
- Representations come from appropriate branching rules for affine Lie algebra
- The coset model

$$\frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1}{\mathfrak{su}(2)_{k+1}} \tag{22}$$

  - Central charge $c = 1 - 6/[(k+2)(k+3)]$
  - Is coset description of unitary minimal models
  - Even non-unitary minimal models if you allow WZW at fractional level (see last section of [dFMS]).

# Modular Invariants

- In CFT one can decompose $\mathcal{H}$ into reps of Virasoro, or really the relevant chiral algebra

$$\mathcal{H} = \bigoplus_{\lambda,\mu} \mathcal{M}_{\lambda,\mu} V_\lambda \otimes V_\mu \tag{23}$$

- The partition function decomposes

$$Z(q) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \tag{24}$$

$$= \sum_{\lambda,\mu} \chi_\lambda(\tau)\, \mathcal{M}_{\lambda,\mu}\, \bar{\chi}_\mu(\bar{\tau}) \tag{25}$$

- Thus *mass matrix* $\mathcal{M}$ classifies combinations of chiral and anti-chiral sectors, and gives multiplicity of primary field labelled by $\lambda$ or $\mu$.

# Modular Invariants: Physical Conditions

$$Z(\tau) = \sum_{\lambda,\mu} \chi_\lambda(\tau)\, \mathcal{M}_{\lambda,\mu}\, \bar{\chi}_\mu(\bar{\tau}) \tag{26}$$

- Recall that the partition function must be modular invariant. i.e. we must have $Z(\tau) = Z(\tau+1) = Z(-1/\tau)$
- Define the $S$ and $T$ matrices by

$$\chi_\lambda(\tau+1) = \sum_\mu T_{\lambda\mu}\chi_\mu(\tau), \qquad \chi_\lambda(-1/\tau) = \sum_\mu S_{\lambda\mu}\chi_\mu(\tau) \tag{27}$$

- This gives us a list of conditions on $M_{\lambda,\mu}$
  1. $\mathcal{M}_{\lambda,\mu}$ is a nonnegative integer.
  2. $\mathcal{M}_{00} = 1$ for a unique vacuum state.
  3. $[\mathcal{M}, S] = [\mathcal{M}, T] = 0$.
- For some theory, what are all the modular invariants?

# MODULAR INVARIANTS: $\mathfrak{su}(2)_k$ AND ADE

- Classification of modular invariants for $\mathfrak{su}(2)_k$ was given by Cappelli, Itzykson, and Zuber... and Kato!
  - Have ADE classification
  - We will give a BS[1] explanation of this classification.
- Start by using finer grained characters which use a chemical potential to track $\mathfrak{su}(2)$ charge

$$\chi_j(z;\tau) = \mathrm{Tr}_{V_j}\left(q^{L_0 - \frac{c}{24}} y^{J_0^3}\right) . \tag{28}$$

- We seek modular invariant partition functions

$$Z(z;\tau) = \sum_{j\bar{j}} \mathcal{M}_{j\bar{j}}\, \chi_j(z;\tau)\bar{\chi}_{\bar{j}}(z;\bar{\tau}) \tag{29}$$

- Note: Things here have $k$ implicit dependency.

[1]Brief Sketch; Bootstrapped; Beautifully Shortened

# Modular Invariants: $\mathfrak{su}(2)_k$ Theta Functions

- We know the partition function can be written in the form

$$Z(z;\tau) = \frac{1}{|\eta(z;\tau)|^2} \sum_{j\bar{j}} R_{j\bar{j}}\, \Theta_j^{(k+2)}(z;\tau)\bar{\Theta}_{\bar{j}}^{(k+2)}(z;\tau) \tag{30}$$

- These are the level-$k$ theta functions

$$\Theta_j^{(k)}(z;\tau) = \sum_{n\in\mathbb{Z}+\frac{m}{2k}} q^{kn^2} q^{kn} \tag{31}$$

- The relationship between $\chi_j(z;\tau)$ and $\Theta_j^{(k)}$ is

$$\chi_j(z;\tau) = \frac{\Theta_{2j+1}^{(k+2)}(z;\tau) - \Theta_{-2j-1}^{(k+2)}(z;\tau)}{\Theta_1^{(2)}(z;\tau) - \Theta_2^{(k+2)}(z;\tau)} \tag{32}$$

# Modular Invariants: $\mathfrak{su}(2)_k$ Character Analysis

- Rather than grind through rep theory, we analyze it after the fact with special function identities:
- The $q$-expansion of the numerator is:

$$q^{\frac{(j+\frac{1}{2})^2}{k+2}}(y^{j+\frac{1}{2}} - y^{-j-\frac{1}{2}}) - q^{\frac{(k-j+\frac{3}{2})^2}{k+2}}(y^{k-j+\frac{3}{2}} - y^{-k+j-\frac{3}{2}}) + \dots \quad (33)$$

- The denominator is: $\Theta_1^{(2)}(z;\tau) - \Theta_2^{(k+2)}(z;\tau)$ expands out to

$$q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \quad (34)$$

- This is amazing! Afterall, we know that

$$\chi_j^{\mathfrak{su}(2)}(z) = \sum_{m=-j}^{j} y^j = \frac{y^{j+\frac{1}{2}} - y^{-j-\frac{1}{2}}}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}} \quad (35)$$

- This means

$$\chi_j(z;\tau) = q^{-\frac{k}{8(k+2)}+\frac{j(j+1)}{k+2}} \frac{\left(\chi_j^{\mathfrak{su}(2)}(z) - q^{k+1-2j}\chi_{k+1-j}^{\mathfrak{su}(2)}(z) + \dots\right)}{\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n)}$$

  - “What else could it be!?”.
  - Just as we would expect, the oscillator modes (the denominator/eta functions) are multiplied by a factor reflecting the states (the numerator), and an overall factor of $q^{h-\frac{c}{24}}$.
  - Moreover, we recognize the subtractions, as subtracting out the null-vector $(J^+_{-1})^{k+1-2j}|j\rangle$ and its descendants, adding back in the descendants that are not there, etc.

# Modular Invariants: $\mathfrak{Vac}_k(2)$ ADE

- Using (generalized) modular transformations and $\Theta$, modular invariant solutions span a $\mathrm{div}(k+2)$-dimensional vector space. Basis vectors

$$Z^{(\alpha,\beta)} = \sum_{j\bar{j}} R^{(\alpha,\beta)}_{j\bar{j}} \Theta^{(k+2)}_j \bar{\Theta}^{(k+2)}_{\bar{j}} \,, \tag{36}$$

$R^{(\alpha,\beta)}_{j\bar{j}} = 1$ if $\alpha|(j+\bar{j})$ and $\beta|(j-\bar{j})$, else 0, and $\alpha\beta = k+2$.

- Writing

$$\mathcal{M}_{j\bar{j}} = \sum_{\substack{\alpha\beta=k+2 \\ \alpha^2<k+2}} C_\alpha \left( R^{(\alpha,\beta)}_{j\bar{j}} - R^{(\beta,\alpha)}_{j\bar{j}} \right) \tag{37}$$

with $C_\alpha \in \mathbb{Z}$, and using consistency properties between the $R^{(\alpha,\beta)}_{j\bar{j}}$ and elementary (but long) divisibility arguments we can arithmetically arrive at a set of solutions to the problem posed.

$$Z_{A_{k+1}} = \sum_{\lambda=1}^{k+1} |\chi_\lambda|^2 \tag{38}$$

$$Z_{D_{2n+2}} = \sum_{\lambda\,\mathrm{odd}}^{2n-1} |\chi_\lambda + \chi_{4n+2-\lambda}|^2 + 2|\chi_{2n+1}|^2 \tag{39}$$

$$Z_{D_{2n+1}} = \sum_{\lambda\,\mathrm{odd}}^{4n-1} |\chi_\lambda|^2 + |\chi_{2n}|^2 + \sum_{\lambda\,\mathrm{even}}^{2n-2} (\chi_\lambda \bar{\chi}_{4n-\lambda} + \mathrm{c.c.}) \tag{40}$$

$$Z_{E_6} = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 \tag{41}$$

$$Z_{E_7} = |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9|^2 + [(\chi_3 + \chi_{15})\bar{\chi}_9 + \mathrm{c.c.}] \tag{42}$$

$$Z_{E_8} = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 \tag{43}$$

# Modular Invariants: $\mathfrak{su}(2)_k$ ADE Data

| Level | Z | Diagram | Coxeter $h$ | Exponents |
|---|---|---|---|---|
| Any $k$ | Eq. (38) | $A_{k+1}$ | $k+2$ | $1, 2, 3, \ldots, k+1$ |
| $k = 4n$ | Eq. (39) | $D_{2n+2}$ | $4n+2$ | $1, 3, 5, \ldots, k+1$ |
| $k = 4n-2$ | Eq. (40) | $D_{2n+1}$ | $4n$ | $1, 3, 5, \ldots, k-1$ |
| $k = 10$ | Eq. (41) | $E_6$ | 12 | $1, 4, 5, 7, 8, 11$ |
| $k = 16$ | Eq. (42) | $E_7$ | 18 | $1, 5, 7, 9, 11, 13, 17$ |
| $k = 28$ | Eq. (43) | $E_8$ | 30 | $1, 7, 11, 13, 17, 19, 23, 29$ |

- Because of coset description of minimal models, minimal models also have ADE classification.
- Modular invariants are in one-to-one correspondence with pairs of Lie algebras $(A, A)$, $(A, D)$, $(D, A)$, $(E, A)$, $(A, E)$.
  - These denote numerator and denominator invariant in coset.
  - e.g. Diagonal modular invariant for $(p, p')$ is

$$Z_{(A_{p'-1}, A_{p-1})} = \sum_{(r,s)\in\mathrm{Kac}} |\chi_{r,s}|^2 \tag{44}$$

  - e.g. If $p' = 2(2n+1)$, then there is a block-diagonal modular invariant for $(p, p')$

$$Z_{(D_{p'/2}, A_{p-1})} = \frac{1}{2} \sum_{\substack{(r,s)\in\mathrm{Kac} \\ r=1 \pmod 2}} |\chi_{r,s} + \chi_{p'-r,s}|^2 \tag{45}$$

- Specializing to unitary case $(p, p') = (m+1, m)$, the modular invariants are of the form
  - $(A_{m-1}, A_m)$, $(A_{m-1}, D_{(m+3)/2})$ if $m$ is odd
  - $(A_{m-1}, A_m)$, $(A_{m-1}, D_{(m+2)/2})$ if $m$ is even
  - Also $(A_{10}, E_6)$, $(E_6, A_{12})$, $(A_{16}, E_7)$, $(E_7, A_{18})$, $(A_{28}, E_8)$, or $(E_8, A_{28})$, if $m = 11,12,17,18,29$, or 30 respectively
- Thus, for each $m$, it makes sense to talk about the $A$-type, $D$-type, or $E$-type model at that $m$ (if they exist).

| m | A-Type Model | D-Type Model |
|---|---|---|
| 3 | Critical Ising | Critical Ising |
| 4 | Tricritical Ising | Tricritical Ising |
| 5 | Tetracritical Ising | Critical 3-State Potts |
| 6 | Pentacritical Ising | Tricritical 3-State Potts |

# Minimal Models and Orbifolds

# Orbifolds: Twisted Sectors

- Suppose we have some theory $T$, on $S^1 \times \mathbb{R}$ for concreteness, with non-anomalous $\mathbb{Z}_2$ symmetry
  - Can couple to background $\mathbb{Z}_2$ connection
  - We have an untwisted Hilbert space $\mathcal{H}_{Un.}$ and twisted Hilbert space $\mathcal{H}_{Tw.}$ depending on if the background $\mathbb{Z}_2$ is trivial. i.e. if we have holonomy around $S^1$.
- $\mathbb{Z}_2$ symmetry acts on the states in these Hilbert spaces, splitting them into $\mathbb{Z}_2$ even and $\mathbb{Z}_2$ odd sectors.

$$\mathcal{H}_{T,\mathrm{Un.}} = \mathcal{H}^{+}_{T,\mathrm{Un.}} \oplus \mathcal{H}^{-}_{T,\mathrm{Un.}} \tag{46}$$

$$\mathcal{H}_{T,\mathrm{Tw.}} = \mathcal{H}^{+}_{T,\mathrm{Tw.}} \oplus \mathcal{H}^{-}_{T,\mathrm{Tw.}} \tag{47}$$

- Gauged/orbifold theory $T//\mathbb{Z}_2$ has both $\mathcal{H}_{T,\mathrm{Un.}}$ and $\mathcal{H}_{T,\mathrm{Tw.}}$ states (sum over all background connections), but only those that are gauge-invariant! Hence

$$\mathcal{H}_{T//\mathbb{Z}_2,\mathrm{Un.}} = \mathcal{H}^{+}_{T,\mathrm{Un.}} \oplus \mathcal{H}^{+}_{T,\mathrm{Tw.}} \tag{48}$$

$$\mathcal{H}_{T//\mathbb{Z}_2,\mathrm{Tw.}} = \mathcal{H}^{-}_{T,\mathrm{Un.}} \oplus \mathcal{H}^{-}_{T,\mathrm{Tw.}} \tag{49}$$

# Orbifolds: Partition Function

- Just as the partition function on $M$ is

$$Z_T(\tau) = \mathrm{Tr}_{\mathcal{H}_{T,Un.}}(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}) \tag{50}$$

- Partition function for orbifold theory can be obtained by summing over all Hilbert spaces with correct projector onto $\mathbb{Z}_2$-invariant states inserted

$$Z_{T//\mathbb{Z}_2}(\tau) = \sum_{h \in \mathbb{Z}_2} \mathrm{Tr}_{\mathcal{H}_{T,h}}(\mathcal{P} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}) \tag{51}$$

$$= \frac{1}{2} \sum_{g,h \in \mathbb{Z}_2} \mathrm{Tr}_{\mathcal{H}_{T,h}}(g q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}) \tag{52}$$

- We can formalize this to

$$Z_{T//\mathbb{Z}_2} = \frac{1}{|H^1(M,G)|} \sum_{\alpha \in H^1(M,G)} Z_T[\alpha]\,. \tag{53}$$

- Twisted $Z_T[\alpha]$ are related by modular transforms

# Orbifolds: $A$-type Twisted $Z$

- $T = (A_{p'-1}, A_{p-1})$, formed by irreps $(r, s)$, where we take $1 \leq r \leq p' - 1$, $1 \leq s \leq p - 1$, and $r + s = 0 \pmod 2$.
- Consider the $\mathbb{Z}_2$ action on the chiral primary $\phi_{(r,s)}$ defined by

$$\phi_{(r,s)} \mapsto (-1)^{r+1} \phi_{(r,s)} \,. \tag{54}$$

- The defining equation of the operator algebra is left invariant

$$\phi_{(r_1,s_1)} \times \phi_{(r_2,s_2)} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 \pmod 2}}^{k=r_1+r_2-1} \; \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=1 \pmod 2}}^{l=s_1+s_2-1} \phi_{(k,l)} \tag{55}$$

- Moreover, this gives us the twisted partition function

$$Z_{+-} := \mathrm{Tr}_{\mathcal{H}}(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}) = \sum_{(r,s)} (-1)^{r+1} |\chi_{r,s}|^2 \tag{56}$$

# ORBIFOLDS: A-TYPE MODULAR BLAH

- The $S$ and $T$ transformations for minimal models are

$$T_{rs;\rho\sigma} = \delta_{r,\rho}\delta_{s,\sigma} e^{2\pi i(h_{r,s} - c/24)} \tag{57}$$

$$S_{rs;\rho\sigma} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma} \sin\left(\pi \frac{p}{p'} r\rho\right) \sin\left(\pi \frac{p'}{p} s\sigma\right) \tag{58}$$

- Allows us to explicitly obtain $Z_{-+} = SZ_{+-}$ and $Z_{--} = TZ_{-+}$.
  - e.g. $Z_{--}$ can be computed vaguely as

$$Z_{--} = \sum_{(r,s)=0} (-1)^{\tilde{p}'+r} \bar{\chi}_{(r,s)} \chi_{\tilde{p}'-r,s} + \sum_{(r,s)=1} (-1)^{\tilde{p}+r} \bar{\chi}_{(r,s)} \chi_{r,\tilde{p}-r-s} \tag{59}$$

- The amazing result is, that after all the work

$$Z_{T//\mathbb{Z}_2} = \frac{1}{2}(Z_T + Z_{+-} + Z_{-+} + Z_{--}) \tag{60}$$

$$= \begin{cases} Z_{(D_{p'/2+1}, A_{p-1})} & p' = 0 \pmod 2 \\ Z_{(A_{p'-1}, D_{p/2+1})} & p = 0 \pmod 2 \end{cases} \tag{61}$$

- In the unitary case, this means the *A* and *D*-type models are exchanged under $\mathbb{Z}_2$ orbifold. KW duality.

| m | A-Type Model | D-Type Model |
|---|---|---|
| 3 | Critical Ising | Critical Ising |
| 4 | Tricritical Ising | Tricritical Ising |
| 5 | Tetracritical Ising | Critical 3-State Potts |
| 6 | Pentacritical Ising | Tricritical 3-State Potts |

- $E_6$ unitary minimal models have $\mathbb{Z}_2$ symmetry, $E_7$ and $E_8$ do not. Also 3-State Potts have enhanced $S_3$ symmetry.
  - Conclude the $E_6$ models must be self-dual under orbifold
- Fermionization?

# Fermionization and Minimal Models

for Quantum Fields and Strings Meeting

Justin Kulp

Perimeter Institute for Theoretical Physics

17/Apr/2020

# Overview

# Motivation

- Dualities are critical in studying physical phenomena.
  - Some theories just have the same IR physics.
  - Some are *exact*. Two different descriptions of same physics. e.g. Kramers-Wannier duality ($g \leftrightarrow g^{-1}$) in Ising.

$$H = -J(\sum_i Z_i Z_{i+1} + g \sum_i X_j) \tag{1}$$

- A particularly interesting type of duality are those relating bosonic and fermionic theories
  - In $d = 2 + 1$, plenty of recent examples in particle-vortex like dualities and Chern-Simons matter theories [AGAY12, JMM$^+$15, KT16, SSWW16].
  - In $d = 1 + 1$ Jordan-Wigner (JW) transform relates 1d spin-$\frac{1}{2}$ chain to theory of fermions. e.g. Ising spin-chain becomes Majorana chain.
  - Can be used to facilitate studies of purely bosonic relationships. e.g. defects in a CFT.

- A point stressed in recent years is that theories related by JW transform should not be thought of as equivalent.
  - Fermionic theories depend on spin-structure, bosonic theories do not
  - In older work, the $c = \frac{1}{2}$ Ising and Majorana fermion CFTs are considered to be the same. But Ising only has integral spin operators, while Majorana fermion has half-integral spin operators.
- Similarly, “smallest unitary $\mathcal{N} = 1$ minimal model” or $c = 7/10$ model, is fermionization of tricritical Ising model
- Since minimal models are so simple, do they all have fermionizations analogous to the Majorana fermion, and smallest $\mathcal{N} = 1$ model?

# Minimal Models

- In classic paper [BPZ84], the authors show that for particular values $h_{r,s} = h_{r,s}(c)$, Verma module $V(c, h_{r,s})$ has null-vector.
  - Modding out and using Ward identities implements constraints on correlation functions. This truncates the OPE (OPE expands into finitely many fields)
  - There may still be an infinite number of families in the theory
- Given positive co-prime integers $p > p' \geq 2$, one gets a *finite number* of local fields with well-defined scaling behaviour if

$$c(p, p') = 1 - 6\frac{(p - p')^2}{pp'} \tag{2}$$

  - Unitary if $(p, p') = (m + 1, m)$.
- With highest weights taking values in

$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'} \tag{3}$$

  - $1 \leq r < p'$ and $1 \leq s < p$.

- To have a full CFT, we need to look at consistent gluings of the chiral halves or *modular invariants* for the theory

$$Z(\tau) = \sum_{(r_1,s_1),(r_2,s_2)} \mathcal{M}_{(r_1,s_1),(r_2,s_2)} \chi_{r_1,s_1}(\tau) \bar{\chi}_{r_2,s_2}(\tau) \tag{4}$$

  1. $\mathcal{M}_{(r_1,s_1),(r_2,s_2)}$ is a nonnegative integer.
  2. $\mathcal{M}_{(1,1),(1,1)} = 1$ for a unique vacuum state.
  3. $[\mathcal{M}, S] = [\mathcal{M}, T] = 0$ for modular invariance.
- To classify the modular invariants for minimal models one "simply" must show if $k = (3p' - 2p)/(p - p')$ then the minimal model with $c(p, p')$ is the WZW coset model

$$\frac{\mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1}{\mathfrak{su}(2)_{k+1}} \tag{5}$$

- Then of course, one must classify the modular invariants of the $\mathfrak{su}(2)_k$ WZW model...

# Minimal Models: ADE Classification

- Modular invariants for $\mathfrak{su}(2)_k$ was given by Cappelli, Itzykson, and Zuber [CIZ87a, CIZ87b] and Kato [Kat87]!
  - Have ADE classification. Coset models inherit this result (after some rep theory)

## Modular Invariants for Minimal Models

Modular invariants of minimal models are in one-to-one correspondence with pairs of simply laced Lie algebras $(A, A)$, $(A, D)$, $(D, A)$, $(E, A)$, $(A, E)$.

- Specializing to unitary case $(p, p') = (m+1, m)$, the modular invariants are of the form
  - $(A_{m-1}, A_m)$, $(A_{m-1}, D_{(m+3)/2})$ if $m$ is odd
  - $(A_{m-1}, A_m)$, $(A_{m-1}, D_{(m+2)/2})$ if $m$ is even
  - Also $(A_{10}, E_6)$, $(E_6, A_{12})$, $(A_{16}, E_7)$, $(E_7, A_{18})$, $(A_{28}, E_8)$, or $(E_8, A_{30})$, if $m = 11,12,17,18,29$, or 30 respectively
- Thus, for each $m$, it makes sense to talk about *the* $A$-type, $D$-type, or $E$-type model at that $m$ (if they exist).

- Take $m = 5$, then $c = 4/5$ and allowed values of $h$ are

| $h_{r,s}$ | $s = 1$ | $s = 2$ | $s = 3$ | $s = 4$ | $s = 5$ |
|---|---|---|---|---|---|
| $r = 1$ | $1(0)$ | $\sigma'(\frac{1}{8})$ | $\epsilon''(\frac{2}{3})$ | $\sigma'''(\frac{13}{8})$ | $\epsilon''''(3)$ |
| $r = 2$ | $\epsilon'(\frac{2}{5})$ | $\sigma(\frac{1}{40})$ | $\epsilon(\frac{1}{15})$ | $\sigma''(\frac{21}{40})$ | $\epsilon'''(\frac{7}{5})$ |

- There is the obvious (diagonal) choice for the theory, giving the partition function $Z_{(A_4,A_5)}(\tau)$. Tetracritical Ising.
- But we also note that there are some off diagonal operators which are bosonic, namely

$$\epsilon'''\bar{\epsilon}' , \quad \epsilon''''\bar{1} , \quad \text{and conjugates.} \tag{6}$$

- Consistency with $S$ and $T$ transformations shows that a perfectly suitable modular invariant partition function is

$$Z_{(A_4,D_4)}(\tau) = \sum_{r=1,2} \left\{ |\chi_{r,1} + \chi_{r,5}|^2 + 2|\chi_{r,3}|^2 \right\} \tag{7}$$

# Modular Invariants: $A$ and $D$ at low $m$

| m | A-Type Model | D-Type Model |
|---|---|---|
| 3 | Critical Ising | Critical Ising |
| 4 | Tricritical Ising | Tricritical Ising |
| 5 | Tetracritical Ising | Critical 3-State Potts |
| 6 | Pentacritical Ising | Tricritical 3-State Potts |

# Orbifolds

- Suppose we have some theory $T$, on $S^1 \times \mathbb{R}$ for concreteness, with non-anomalous $\mathbb{Z}_2$ symmetry
  - Can couple to background $\mathbb{Z}_2$ connection. Gives untwisted $\mathcal{H}_{Un.}$ and twisted Hilbert space $\mathcal{H}_{Tw.}$.
- $\mathbb{Z}_2$ symmetry splits Hilbert spaces into even and odd sectors.

$$\mathcal{H}_{T,\mathrm{Un.}} = \mathcal{H}^{+}_{T,\mathrm{Un.}} \oplus \mathcal{H}^{-}_{T,\mathrm{Un.}} \tag{8}$$

$$\mathcal{H}_{T,\mathrm{Tw.}} = \mathcal{H}^{+}_{T,\mathrm{Tw.}} \oplus \mathcal{H}^{-}_{T,\mathrm{Tw.}} \tag{9}$$

- Gauged theory $T//\mathbb{Z}_2$ has both $\mathcal{H}_{T,\mathrm{Un.}}$ and $\mathcal{H}_{T,\mathrm{Tw.}}$ states, but only those that are gauge-invariant!

$$\mathcal{H}_{T//\mathbb{Z}_2,\mathrm{Un.}} = \mathcal{H}^{+}_{T,\mathrm{Un.}} \oplus \mathcal{H}^{+}_{T,\mathrm{Tw.}} \tag{10}$$

$$\mathcal{H}_{T//\mathbb{Z}_2,\mathrm{Tw.}} = \mathcal{H}^{-}_{T,\mathrm{Un.}} \oplus \mathcal{H}^{-}_{T,\mathrm{Tw.}} \tag{11}$$

- $T//\mathbb{Z}_2$ has a new $\hat{\mathbb{Z}}_2$ symmetry coming from the even/odd splitting, and $T//\mathbb{Z}_2//\hat{\mathbb{Z}}_2 = T$.

- In terms of twisted partition functions

$$Z_T[\alpha_1, \alpha_2] = \mathrm{Tr}_{\mathcal{H}_{T,\alpha_1}}(\alpha_2 \cdot q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}) \tag{12}$$

this is simply saying

$$Z_{T//G}[\beta_1, \beta_2] = \frac{1}{\sqrt{|H^1(M, G)|}} \sum_{\alpha \in H^1(M,G)} (-1)^{\int \alpha \cup \beta} Z_T[\alpha] \,. \tag{13}$$

- Consider the diagonal minimal model $T = (A_{p'-1}, A_{p-1})$. There is a symmetry transformation

$$\phi_{(r,s)} \mapsto (-1)^{(m+1)r+ms+1}\phi_{(r,s)}\,. \tag{14}$$

- This gives us the twisted partition function with non-trivial defect around space direction

$$Z_T[0,1] = \sum_{(r,s)}(-1)^{(m+1)r+ms+1}|\chi_{r,s}|^2 \tag{15}$$

  - $S$ and $T$ transformations give $Z_T[1,0]$ and $Z_T[1,1]$.
- The amazing result is, that after all the work

$$\begin{aligned} Z_{T//\mathbb{Z}_2} &= \frac{1}{2}(Z_T[0,0] + Z_T[0,1] + Z_T[1,0] + Z_T[1,1]) \\ &= \begin{cases} Z_{(D_{p'/2+1},A_{p-1})} & p' = 0 \pmod 2 \\ Z_{(A_{p'-1},D_{p/2+1})} & p = 0 \pmod 2 \end{cases} \end{aligned} \tag{16}$$

# Orbifolds: Unitary Minimal Models

- In the unitary case, this means *the A* and *D*-type models are exchanged under $\mathbb{Z}_2$ orbifold.

| m | A-Type Model | D-Type Model |
|---|---|---|
| 3 | Critical Ising | Critical Ising |
| 4 | Tricritical Ising | Tricritical Ising |
| 5 | Tetracritical Ising | Critical 3-State Potts |
| 6 | Pentacritical Ising | Tricritical 3-State Potts |

- $E_6$ unitary minimal models have $\mathbb{Z}_2$ symmetry, $E_7$ and $E_8$ do not. Also 3-State Potts have enhanced $S_3$ symmetry [RV98, CW20].
  - Conclude the $E_6$ models must be self-dual under orbifold
  - These are the symmetries of the associated Dynkin diagrams.
  - Symmetry of RSOS model not enhanced or broken in continuum limit.

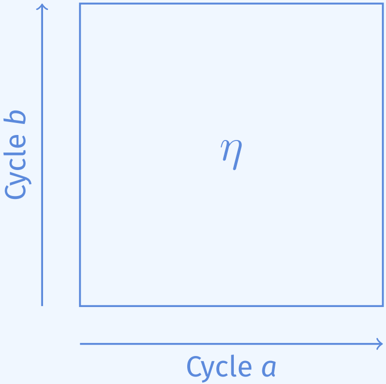
# Fermionization

- By a 2d fermionic theory, I mean a QFT that requires spin structure and possibly has operators of half-integral spin
  - Comes equipped with a $\mathbb{Z}_2^f$ or $(-1)^F$ symmetry
  - "Background connection" is spin-structure, an "affine $\mathbb{Z}_2$ connection" with curvature $w_2(M)$
- Two topological invertible phases for 2d theory with $(-1)^F$
  - Topological invertible phase given by $\mathrm{Hom}(\Omega_2^{\mathrm{Spin}}(pt), U(1))$ [KTTW15]. Isomorphic to $\mathbb{Z}_2$, non-trivial element $(-1)^{\mathrm{Arf}[\eta]}$
  - Two phases can be thought of as 2d analog for 3d Chern-Simons term.

$$Z_{\mathrm{Maj.}}(m \gg 0, \eta)/Z_{\mathrm{Maj.}}(m \ll 0, \eta) = (-1)^{\mathrm{Arf}[\eta]} \tag{17}$$

- Practically $Arf[\eta] = \{\mathrm{Number\,of\,Zero\,Modes}\} \mod 2$
- Alternatively, these are the two phases of the Kitaev Majorana chain [Kit01, Kit09, FK11].

# Fermionization: Quadratic Refinement




If a fermion goes around cycle $a$

$$L_f(a)\,|f,\eta\rangle = (-1)^{q_\eta(a)}\,|f,\eta\rangle$$

where

$$q_\eta(a) = \begin{cases} 0 & \text{if NS} \\ 1 & \text{if R} \end{cases}$$

Then we see that

$$q_\eta(a) + q_\eta(b) = q_\eta(a+b) + a \cap b \tag{18}$$

# Fermionization: Fermionic Partition Function

- Result is that there is a bijection between spin-structures $\eta$ on $M$ and quadratic refinements $q_\eta : H^1(M, \mathbb{Z}_2) \to \mathbb{Z}_2$.
  - Associates a spin-structure to a function on $\mathbb{Z}_2$ gauge fields
- The partition function for a fermionic theory $T_f$ is a discrete Fourier transform of $T_b$. This is a generalized *Jordan-Wigner* transformation

$$Z_{T_f}[\eta] = \frac{1}{\sqrt{|H^1(M, \mathbb{Z}_2)|}} \sum_{\alpha} (-1)^{q_\eta(\alpha)} Z_{T_b}[\alpha] \tag{19}$$

- Moreover, if we compare Hilbert spaces for $T_b$ and $T_f$

$$\mathcal{H}^+_{NS} = \mathcal{H}^+_{\mathrm{Un.}} \quad \mathcal{H}^-_{NS} = \mathcal{H}^+_{\mathrm{Tw.}} \tag{20}$$

$$\mathcal{H}^+_{R} = \mathcal{H}^-_{\mathrm{Un.}} \quad \mathcal{H}^-_{R} = \mathcal{H}^-_{\mathrm{Tw.}} \tag{21}$$

# Fermionization: GSO Projection

- The reverse operation is a GSO projection

$$Z_{T_b}[\alpha] = \frac{1}{\sqrt{|H^1(M, \mathbb{Z}_2)|}} \sum_\eta (-1)^{q_\eta(\alpha)} Z_{T_f}[\eta] \tag{22}$$

- Stacking a fermionic theory $T_f$ with $\mathrm{Arf}$ flips the fermion parity operator for the Ramond states.
  - This swaps $\mathcal{H}_R^+ \longleftrightarrow \mathcal{H}_R^-$.
  - Bosonically, this is $\mathcal{H}^-_{\mathrm{Un.}} \longleftrightarrow \mathcal{H}^+_{\mathrm{Tw.}}$
- This means that

$$\mathrm{GSO}[T_f] // \mathbb{Z}_2 = \mathrm{GSO}[T_f \times \mathrm{Arf}] \tag{23}$$

  - This can be shown easily/algebraically from the definitions we have so far.
  - From Kitaev Majorana chain perspective, stacking with $\mathrm{Arf}$ changes whether we are studying the Kitaev chain with $m < 0$ (unpaired) or $m > 0$ (paired) edge modes.

# Fermionization: Pictorial Explanation

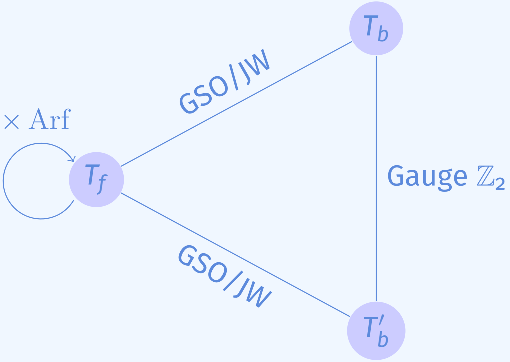



**Figure:** Gauging the $(-1)^F$ symmetry of a spin theory $T_f$ produces a bosonic theory $T_b$ with $\mathbb{Z}_2$ symmetry. A different bosonic theory $T_b'$ is produced if one first stacks with the $\mathrm{Arf}$ theory. These two theories are related by $\mathbb{Z}_2$ orbifold. Since the two fermionic minimal models differ only in assignment of $(-1)^F$ in $\mathrm{R}$-sector, we choose not to distinguish them.

# Fermionic Minimal Models

# Fermionic Minimal Models: Ising and Majorana

- This formula tells us the familiar relationship between Majorana and Ising

$$Z_{\mathrm{Maj.}}[\mathrm{NS},\mathrm{NS}] = \chi_0\bar{\chi}_0 + \chi_0\bar{\chi}_{\frac{1}{2}} + \chi_{\frac{1}{2}}\bar{\chi}_0 + \chi_{\frac{1}{2}}\bar{\chi}_{\frac{1}{2}} \tag{24}$$

$$Z_{\mathrm{Maj.}}[\mathrm{NS},\mathrm{R}] = \chi_0\bar{\chi}_0 - \chi_0\bar{\chi}_{\frac{1}{2}} - \chi_{\frac{1}{2}}\bar{\chi}_0 + \chi_{\frac{1}{2}}\bar{\chi}_{\frac{1}{2}} \tag{25}$$

$$Z_{\mathrm{Maj.}}[\mathrm{R},\mathrm{NS}] = 2\chi_{\frac{1}{16}}\bar{\chi}_{\frac{1}{16}} \tag{26}$$

$$Z_{\mathrm{Maj.}}[\mathrm{R},\mathrm{R}] = 0 \tag{27}$$

  - GS in R sector is degenerate $\left|\frac{1}{16}\right\rangle_{\pm}$, with different signs under $(-1)^F$ so $\mathrm{Tr}_{\mathrm{R}}(-1)^F q^{L_0} = 0$. Stacking with $\mathrm{Arf}$ swaps the $(-1)^F$ assignments.
  - Ising model is self-dual under orbifold, so we have Ramond-Ramond is 0.

- With all of these generalities under wraps, we can obtain some results by stringing together specifics.
- In [arXiv:2002.12283] use the fact that there is a $\mathbb{Z}_2$ orbifold relating $A$ and $D$ type unitary minimal models. The authors argue that there is a fermionic model (spin-structure, half-integral spin operators) for each $(A, A) \leftrightarrow (A, D)$ pair of minimal models.

| m | A-Type Model | D-Type Model | Fermionic Model |
|---|---|---|---|
| 3 | Critical Ising | Critical Ising | Majorana Fermion |
| 4 | Tricritical Ising | Tricritical Ising | $\mathcal{N} = 1$ S-Minimal Model |
| 5 | Tetracritical Ising | Critical 3-State Potts | $m = 5$ Fermionization |
| 6 | Pentacritical Ising | Tricritical 3-State Potts | $m = 6$ Fermionization |

- Some results similar to this had already been obtained in [arXiv:2001.05055].
  - However, the authors were particularly interested in fermionic CFT defects.
  - In particular, they only studied minimal models where the chiral algebra could be extended by an operator of half-integral spin.
  - This restricted their study of (unitary) models to just those with $m \equiv 0, 3 \mod 4$, because these have a W-generator with half-integer spin (as opposed to $m \equiv 1, 2$ which have W-generator with integral spin).

# Fermionic Minimal Models: Some Papers 3/4

- These fermionic models have spin-chain realizations.
  - Take spin-chain for bosonic minimal model. Sites may have state-space $\mathbb{C}^k$ with $k \neq 2$, so embed $\mathbb{C}^k \subseteq (\mathbb{C}^2)^{\otimes \ell}$ so sites of the original chain are unit cells of $\ell$ sites in new chain.
  - Remove the additional $2^\ell - k$ unnecessary states by adding large (local) terms to the Hamiltonian
  - Apply usual JW transformation on spin-$\frac{1}{2}$ chain
- Consider the 3-state Potts. Basis $\{|n\rangle\}_{n=0,1,2}$ with $Z : |n\rangle \mapsto \omega^n |n\rangle$ and $X : |n\rangle \mapsto |n+1\rangle$ the Hamiltonian is

$$H = -J\left(\sum_i Z_i Z_{i+1}^{-1} + g \sum_j X_j\right) \tag{28}$$

- Get spin-$\frac{1}{2}$ setup by replacing

$$|0\rangle_i = |\uparrow\rangle_{2i-1} |\uparrow\rangle_{2i} \qquad |1\rangle_i = \frac{1}{\sqrt{2}} |\downarrow\rangle_{2i-1} (-|\uparrow\rangle_{2i} + |\downarrow\rangle_{2i})$$

$$|2\rangle_i = \frac{1}{\sqrt{2}} |\downarrow\rangle_{2i-1} (|\uparrow\rangle_{2i} + |\downarrow\rangle_{2i}) \qquad |Aux\rangle = |\uparrow\rangle_{2i-1} |\downarrow\rangle_{2i}$$

- In a short note [arXiv:2003.04278] point out that it is known that $E_6$ unitary minimal models have $\mathbb{Z}_2$ symmetry, $E_7$ and $E_8$ do not.
  - Conclude that the $m = 11$ and $m = 12$ exceptional unitary minimal models, $(A_{10}, E_6)$ and $(E_6, A_{12})$, have fermionizations.
  - Moreover, from the symmetry result, there are no more $\mathbb{Z}_2$s left to fermionize. So these must be the only missing cases.
  - Said in the converse, if there was another unitary fermionic theory, a GSO projection should give a unitary bosonic theory with $c < 1$ and non-anomalous $\mathbb{Z}_2$.
- We can check this result too, if we calculate the partition functions we should find (as argued before) that
  1. There is an "honest" Kramers-Wannier duality for the $E_6$ models. i.e. they are self-dual under orbifold.
  2. The Ramond-Ramond partition functions should be 0.

$$Z[0,0] = \sum_{r=1,\mathrm{odd}}^{10} \left|\chi_{(r,1)} + \chi_{(r,7)}\right|^2 + \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 + \left|\chi_{(r,5)} + \chi_{(r,11)}\right|^2$$

$$Z[0,1] = \sum_{r=1,\mathrm{odd}}^{10} \left|\chi_{(r,1)} + \chi_{(r,7)}\right|^2 - \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 + \left|\chi_{(r,5)} + \chi_{(r,11)}\right|^2$$

$$Z[1,0] = \sum_{r=1,\mathrm{odd}}^{10} \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 + \left\{(\chi_{(r,1)} + \chi_{(r,7)})^*(\chi_{(r,5)} + \chi_{(r,11)}) + \mathrm{c.c.}\right\}$$

$$Z[1,1] = \sum_{r=1,\mathrm{odd}}^{10} \left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2 - \left\{(\chi_{(r,1)} + \chi_{(r,7)})^*(\chi_{(r,5)} + \chi_{(r,11)}) + \mathrm{c.c.}\right\}$$

# Fermionic Minimal Models: The Result $(A_{10}, E_6)$

$$Z_f[0,0] = \sum_{r=1,\mathrm{odd}}^{10} \left|\chi_{(r,1)} + \chi_{(r,7)}\right|^2 + \left|\chi_{(r,5)} + \chi_{(r,11)}\right|^2 + \left\{\left(\chi_{(r,1)} + \chi_{(r,7)}\right)^* \left(\chi_{(r,5)} + \chi_{(r,11)}\right) + \mathrm{c.c.}\right\}$$

$$Z_f[0,1] = \sum_{r=1,\mathrm{odd}}^{10} \left|\chi_{(r,1)} + \chi_{(r,7)}\right|^2 + \left|\chi_{(r,5)} + \chi_{(r,11)}\right|^2 - \left\{\left(\chi_{(r,1)} + \chi_{(r,7)}\right)^* \left(\chi_{(r,5)} + \chi_{(r,11)}\right) + \mathrm{c.c.}\right\}$$

$$Z_f[1,0] = \sum_{r=1,\mathrm{odd}}^{10} 2\left|\chi_{(r,4)} + \chi_{(r,8)}\right|^2$$

$$Z_f[1,1] = 0$$

# Conclusion

# Conclusion: Summary and Future Direction

**Summary:**

- Recent advances in condensed matter and high energy physics have provided a sharper understanding of the relationship between bosonic and fermionic theories and orbifolds
- The most tractable CFTs are the minimal models, the fermionic analogs of all minimal model CFTs are now well explored at the level of: duality defects, partition functions, and explicit lattice realizations

**Future:**

- In [CW20] they compute global symmetries for $SU(2)_k$ CFTs, so all of these explicit calculations could be extended to anything written in terms of $SU(2)_k$ WZW models.
- It would be nice to understand RSOS models further

# References I


- Ofer Aharony, Guy Gur-Ari, and Ran Yacoby, *Correlation functions of large n chern-simons-matter theories and bosonization in three dimensions*, Journal of High Energy Physics **2012** (2012), no. 12.
- A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Physics B **241** (1984), no. 2, 333 – 380.
- A. Cappelli, C. Itzykson, and J.-B. Zuber, *Modular Invariant Partition Functions in Two Dimensions*, Nuclear Physics B **280** (1987), 445 – 465.
- Andrea Cappelli, Claude Itzykson, and Jean-Bernard Zuber, *The A-D-E Classification of Minimal and $A_1^{(1)}$ Conformal Invariant Theories*, Communications in Mathematical Physics **113** (1987), no. 1, 1–26.
- Meng Cheng and Dominic J. Williamson, *Relative anomaly in (1+1)d rational conformal field theory*, 2020.

# References II


- Lukasz Fidkowski and Alexei Kitaev, *Topological phases of fermions in one dimension*, Physical Review B **83** (2011), no. 7.
- Sachin Jain, Mangesh Mandlik, Shiraz Minwalla, Tomohisa Takimi, Spenta R. Wadia, and Shuichi Yokoyama, *Unitarity, crossing symmetry and duality of the s-matrix in large n chern-simons theories with fundamental matter*, Journal of High Energy Physics **2015** (2015), no. 4.
- Akishi Kato, *Classification of Modular Invariant Partition Functions in Two-dimensions*, Mod. Phys. Lett. A **2** (1987), 585.
- A Yu Kitaev, *Unpaired majorana fermions in quantum wires*, Physics-Uspekhi **44** (2001), no. 10S, 131–136.
- Alexei Kitaev, *Periodic table for topological insulators and superconductors*, AIP Conference Proceedings (2009).
- Andreas Karch and David Tong, *Particle-vortex duality from 3d bosonization*, Physical Review X **6** (2016), no. 3.

# References III


- Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang, *Fermionic symmetry protected topological phases and cobordisms*, Journal of High Energy Physics **2015** (2015), no. 12, 1–21.
- P. Ruelle and O. Verhoeven, *Discrete symmetries of unitary minimal conformal theories*, Nuclear Physics B **535** (1998), no. 3, 650–680.
- Nathan Seiberg, T. Senthil, Chong Wang, and Edward Witten, *A duality web in 2+1 dimensions and condensed matter physics*, Annals of Physics **374** (2016), 395–433.