Orbifold Groupoids

for CMSA Quantum Matter Seminar

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Overview

Introduction

Introduction: Why Orbifolds? 1/2

- **Orbifolds** are ubiquitous in physics, explicitly and implicitly.
  - Explicitly: CFT and string theory, studying space of CFTs and pheno ['80s-'00s]. E.g. σ models, $c = 1$ conformal manifold.
  - Implicitly: HEP, statistical physics, condensed matter; really orbifolding is just **gauging a finite symmetry** group in (1+1)d.
- **Kramers-Wannier duality** for Ising spin-chain

$$H = -J \sum_i (Z_i Z_{i+1} + g X_i) \quad (1)$$

  - Map to new Ising spins on the links of spin chain: $\tilde{X}_{i+\frac{1}{2}} = Z_i Z_{i+1}$ and $\tilde{Z}_{i-\frac{1}{2}} \tilde{Z}_{i+\frac{1}{2}} = X_i$. Then $g \leftrightarrow g^{-1}$.
  - Example of a duality relating strong and weak couplings
  - $\mathbb{Z}_2$ orbifold of Ising model in high-temperature phase is Ising in low temperature phase [Kramers, Wannier '41], [Kogut '79], [Frölich, Fuchs, Runkel, Schweigert '05], [Radičević '18], [More++] + [Polchinski '14].

Introduction: Why Orbifolds? 2/2

- Familiar **fermionization** and **bosonization** relationships come from orbifolds.
  - In Ising chain, can replace spin variables with fermionic creation/annihilation operators to produce Kitaev-Majorana chain [Jordan, Wigner ’28].
  - Free Majorana fermion is dual to Ising model in (1+1)d. Free Dirac fermion is dual to compact boson [Thorngren ’18], [Ji, Shao, Wen ’19], [Karch, Tong, Turner ’19], [More++].
- In this case, rather than gauging internal symmetry, we are gauging fermion parity $(-1)^F$, i.e. **GSO Projection**.
  - This amounts to summing over spin structures [Seiberg, Witten ’86], [Gaiotto, Kapustin ’15], [Kaidi, Parra-Martinez, Tachikawa, Debray ’19].
  - We call inverse process a **Jordan-Wigner transformation**.

Introduction: Plans and Not-Plan

Question: *Given some (1+1)d QFT, how many new theories can we produce by successive orbifolds?*

Plan:

1. Generalities of orbifolds in (1+1)d QFTs and other "topological manipulations" for bosonic theories.
2. Couple to (2+1)d topological field theories. Use TFT machinery to probe data of QFT.
3. Bosonic examples: KW Duality is EM Duality. $\mathbb{Z}_2 \times \mathbb{Z}_2$.
4. Short fermionic story
5. Fermionic examples: KW Duality, Jordan Wigner, and GSO.

Disclaimer: Ground up. Keep CFT examples in mind. Take groups to be finite Abelian. Keep things oriented. Keep things (1+1)d. 0-form symmetries. No RG stuff.

Bosonic Orbifolds

Bosonic Orbifolds: Twisted Partition Functions

- Consider a (1+1)d theory T with finite abelian A symmetry.
  - Coupling theory to background A-connection gives different **twisted partition functions** for T, $Z_T[\alpha]$. $\alpha \in H^1(M, A)$ labels holonomies for noncontractible loops.
  - If A-symmetry is **non-anomalous**, i.e. bkgd connections above were well-defined, then we gauge by summing over all α.
  - Emergent $\hat{A}$ symmetry [Vafa '89], [Tachikawa '17], with connection labelled by $\beta \in H^1(M, \hat{A})$.

$$Z_{[T/A]}[\beta] = \frac{1}{|A|} \sum_{\alpha} e^{i \int \alpha \cup \beta} Z_T[\alpha] \tag{2}$$

- Can also add choice of **discrete torsion**; $U(1)$ weight $\epsilon_{\nu_2}(\alpha)$ for the twisted sectors. $\nu_2 \in H^2(A, U(1))$. [Vafa '86]

$$Z_{[T/_{\nu_2} A]}[\beta] = \frac{1}{|A|} \sum_{\alpha} e^{i \int \alpha \cup \beta} \epsilon_{\nu_2}(\alpha) Z_T[\alpha] \tag{3}$$

Bosonic Orbifolds: Twisted Partition Functions

- Can view discrete torsion as stacking T with **(1+1)d G-SPT** $\epsilon_{\nu_2} \sim e^{iS_{\nu_2}}$, before gauging [Chen, Gu, Liu, Wen '11], [Kapustin '14].
  - Mathematically, we have to pick a concrete 2-cocycle trivializing the A 't Hooft anomaly (or lack thereof).

$$Z_{[T/_{\nu_2}A]}[\beta] = \frac{1}{|A|} \sum_{\alpha} e^{i\int \alpha \cup \beta} \epsilon_{\nu_2}(\alpha) Z_T[\alpha] \tag{4}$$

- More broadly, stacking a theory with such a topological phase changes Zs, but otherwise doesn't talk to original DoF encoded in T.

- Orbifolding also leaves local dynamics unchanged. Changes Zs and correlation functions on topologically non-trivial manifolds.

Bosonic Orbifolds: Topological Manipulations

- Leads to idea of **topological manipulations** of T. Leave theory unchanged at level of local dynamics, but different global properties, e.g. Z and $\langle \cdots \rangle$ on non-trivial topology.
  - Coupling to (dynamical) discrete gauge field
  - Stacking with (1+1)d SPT phase

Example: $\mathfrak{su}(2)_k$ WZW and Modular Invariants

Full RCFT requires choice of **modular invariant**, not just chiral algebra. $\mathfrak{su}(2)_k$ WZW modular invariants have ADE classification [Cappelli, Ityzkson, Zuber '87], [Kato '87]. If k is even, the A-type model has non-anomalous $\mathbb{Z}_2$ symmetry; orbifold to produce D-type model.

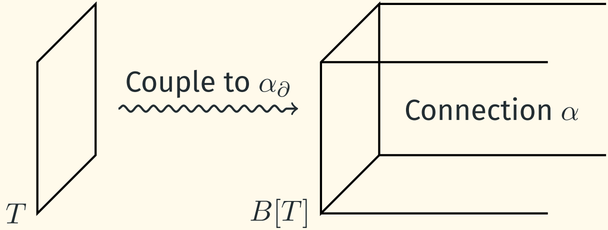
- Want to study topological manipulations of T, so lets **decouple** the local dynamics of T from topological data.

(2+1)d Gauge Theory

(2+1)d Gauge Theory: Boundary Conditions 1/2

- There is a **bijection** between (1+1)d theories with A-symmetry (and anomaly μ_3) and boundary conditions for topological (2+1)d A gauge theory. $\mathrm{DW}[A]_{\mu_3}$ [Dijkgraaf, Witten '90].

$\Rightarrow$ Coupling T to boundary value of dynamical A connection produces **enriched Neumann boundary condition** $B[T]$.

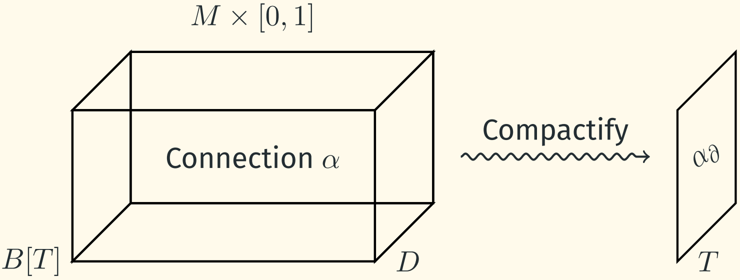


$$|T\rangle = \sum_\alpha Z_T[\alpha]\,|\alpha\rangle \quad (5)$$

(2+1)d Gauge Theory: Boundary Conditions 2/2

$\Leftarrow$ Placing $B[T]$ on one side of a (2+1)d slab and **Dirichlet boundary conditions** on the other recovers T.

- The Dirichlet boundary is endowed with the **global** A-symmetry of T.



$$\langle T | D[\alpha_\partial] \rangle = Z_T[\alpha_\partial] \tag{6}$$

(2+1)d Gauge Theory: Topological Boundaries

- We have effectively blown up T to a topological sandwich.
  - All **dynamics** of T lives **on one side** in an enriched Neumann boundary condition $B[T]$.
  - **Topological** information is contained in the topological boundary condition **on the opposite side**.
- By construction, topological manipulations of T involving A **only** affect the Dirichlet boundary condition, not $B[T]$.
  - E.x. The orbifold theory $[T/_{\nu_2}A]$ is obtained from having the "totally Neumann boundary condition" on one side.

$$\sum_{\beta} \langle T|\beta\rangle = \sum_{\alpha,\beta} Z_T[\alpha] \langle \alpha|\beta\rangle \tag{7}$$

$$= \frac{1}{|A|} \sum_{\alpha} Z_T[\alpha] \tag{8}$$

$$= Z_{[T/_{\nu_2}A]} \tag{9}$$

(2+1)d Gauge Theory: Dual Descriptions

- More generally, we can replace D by any (irreducible bosonic)† **topological boundary condition** for the A gauge theory with action μ_3.
  - Classified by **pairs** (H, ν_2) where $H \leq A$ and $\nu_2 \in H^2(H, U(1))$ with $\delta\nu_2 = \mu_3|_H$ [Ostrik '03].
  - i.e. a choice of Neumann subgroup and boundary action.
- Is $[T/_{\nu_2} A]$ a slab of $\hat{A}$ gauge theory with Dirichlet BCs? Or A gauge theory with Neumann BCs? Both.
  - A and $\hat{A}$ gauge theory are dual descriptions of same *abstract* TFT. (2+1)d TFT is captured by MTC $\mathcal{C}$ of anyons.
- Discrete gauge theory is just some Wilson lines, labelled by irreps of A, some disorder defects carrying flux, and some fusion products.
  - It's a gauge theoretic realization of some quantum double.
- Succinctly: who is the Wilson line?

Example: Realizations of Toric Code

Toric code has anyons $\{1, e, m, f\}$. Also has two topological boundary conditions, B_e and B_m, upon which e and m lines can terminate resp. [Kitaev '97], [Bravyi, Kitaev '98], [Kitaev, Kong '12].

In $\mathbb{Z}_2$ gauge theory realization, one of e or m is the "Wilson line," while the other is an "'t Hooft line." Also, boundary defects at a Dirichlet BC are disorder defects.

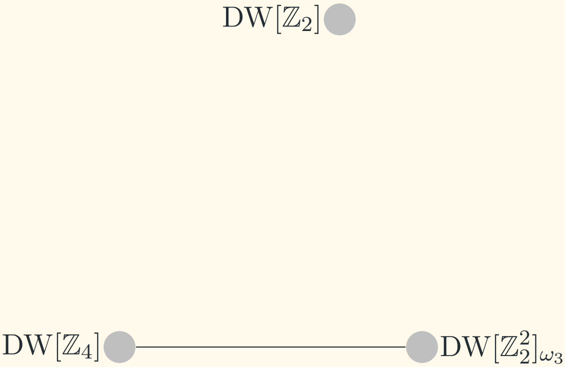
So if e is the Wilson line, then the boundary condition B_e (populated by $\{1, m\}$) looks like a Dirichlet BC. But if m is the Wilson line, then this looks like Neumann BC.

- Realize toric code as $\mathrm{DW}[\mathbb{Z}_2]$ theory by presenting a topological BC (e.g. B_e) with boundary anyons (e.g. $\{1, m\}$) which fuse with $\mathbb{Z}_2$ group law.

(2+1)d Gauge Theory: Duality Groupoids

- Can build a **duality groupoid**, graph whose vertices are DW theories and edges are isomorphisms of (2+1)d TFTs.
  - Edges include **symmetries** of DW theory. E.x. $\mathbb{Z}_2$ gauge theory has $\mathbb{Z}_2$ symmetry exchanging e and m lines.
  - Includes **identifications** of $\mathrm{DW}[A]_{\mu_3}$ with distinct $\mathrm{DW}[A']_{\mu_3'}$. E.x. $\mathrm{DW}[\mathbb{Z}_4]$ and $\mathrm{DW}[\mathbb{Z}_2^2]_{\omega_3}$ [Wang, Wen, Witten '17], [Tachikawa '17].
- Any isomorphism of TFTs turning $\mathrm{DW}[A]_{\mu_3}$ into $\mathrm{DW}[A']_{\mu_3'}$ maps not just anyons, but also boundary conditions.
  - Dirichlet boundary conditions must always be mapped to some (H', ν_2') with global A symmetry.
- On the Brauer groups of symmetries of Abelian Dijkgraaf-Witten Theories [Fuchs, Priel, Schweigert, Valentino '15].
  - *Universal Kinematical*. Automorphisms of A.
  - *Universal Dynamical*. Symmetries of topological action.
  - *EM Dualities*. Interchanging A and $\hat{A}$.

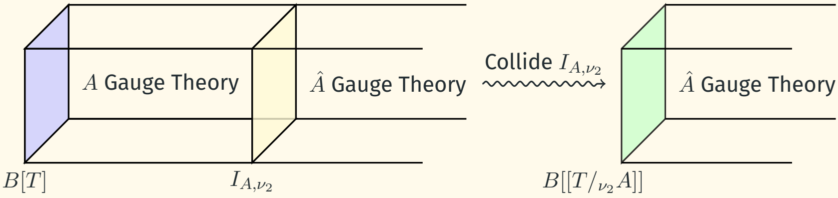
(2+1)d Gauge Theory: Duality Groupoid Examples



(2+1)d Gauge Theory: Orbifold Groupoids

- Combine these maps between (2+1)d TFTs and BCs with our bijection between (1+1)d theories and BCs.
  - Each **TFT map**/symmetry/duality is an irreducible **topological manipulation** of the (1+1)d theory!
- Some topological manipulations don't really change the (1+1)d theory, only how it's coupled to flat connection.
  - From (2+1)d perspective, these are the symmetries of DW fixing Dirichlet BCs. i.e. those *not* of EM Duality type.
- Introduce **orbifold groupoids**. A refinement of the duality groupoids, with two objects being distinct if they are related by an EM-like duality.
  - This answers: how many new theories can we produce?

- If there is a duality between two theories $\mathrm{DW}[A]_{\mu_3}$ and $\mathrm{DW}[A']_{\mu_3'}$, we can construct an invertible **duality interface** between the two.
  - Use duality interface as a tool to implement the dualities between theories on their inhabitants.
- View orbifold $T \mapsto [T/_{\nu_2} A]$ as an operation on the boundary conditions $B[T] \mapsto B[[T/_{\nu_2} A]]$.
  - Physically, $B[[T/_{\nu_2} A]]$ is obtained by colliding the **orbifold interface** I_{A,ν_2} with $B[T]$.



Bxamples

Bxamples: KW Duality is EM Duality 1/2

- Suppose we have a (1+1)d theory T with non-anomalous $\mathbb{Z}_2$ symmetry.
  - No non-trivial automorphisms of $\mathbb{Z}_2$.
  - No SPT phases which can be stacked with theory.
  - All we can do is orbifold the full $\mathbb{Z}_2$.
- On torus, in basis of $\mathbb{Z}_2$ holonomies

$$Z_{[T/\mathbb{Z}_2]}[\beta_1, \beta_2] = \frac{1}{2} \sum_{\alpha_1, \alpha_2} (-1)^{\alpha_1 \beta_2 - \alpha_2 \beta_1} Z_T[\alpha_1, \alpha_2] \,, \tag{10}$$

- The orbifold operation lifts to non-trivial interface in $\mathbb{Z}_2$ gauge theory.
  - Collides with boundary theory described by $Z_T[\alpha]$ to produce boundary theory described by $Z_{[T/\mathbb{Z}_2]}[\beta]$.

$$I_{\text{gauge}}[\alpha_1, \alpha_2; \beta_1, \beta_2] = (-1)^{\alpha_1 \beta_2 - \alpha_2 \beta_1} \,. \tag{11}$$

Bxamples: KW Duality is EM Duality 2/2

- This is a nice basis for thinking about $\mathbb{Z}_2$ twists of a (1+1)d theory, but not for dealing with (2+1)d TFT.
  - Better basis to use is labelled by anyons of toric code.
- We identify the anyons in the toric code with the linear combinations

$$\hat{Z}[0,\hat{0}] = \frac{1}{2}(Z[0,0] + Z[0,1]) = Z_1 \tag{12}$$

$$\hat{Z}[0,\hat{1}] = \frac{1}{2}(Z[0,0] - Z[0,1]) = Z_e \tag{13}$$

$$\hat{Z}[1,\hat{0}] = \frac{1}{2}(Z[1,0] + Z[1,1]) = Z_m \tag{14}$$

$$\hat{Z}[1,\hat{1}] = \frac{1}{2}(Z[1,0] - Z[1,1]) = Z_f\,. \tag{15}$$

- In this basis

$$\hat{I}_{\text{gauge}}[\alpha_1, \hat{\alpha}_2; \beta_1, \hat{\beta}_2] = \delta_{\hat{\alpha}_2\beta_1}\delta_{\hat{\beta}_2\alpha_1}\ ! \tag{16}$$

(1+1)d Perspective

- Suppose we have a (1+1)d theory T with non-anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.
  - A $GL(2;\mathbb{F}_2) \cong S_3$ of automorphisms of $\mathbb{Z}_2^2$.
  - An $H^2(\mathbb{Z}_2^2, U(1)) = \mathbb{Z}_2$ of (1+1)d SPT phases to stack. E.x.

$$Z[\alpha_a, \alpha_b, \beta_a, \beta_b] \mapsto (-1)^{\ell(\alpha_a\beta_b - \alpha_b\beta_a)} Z[\alpha_a, \alpha_b, \beta_a, \beta_b]\,, \qquad (17)$$

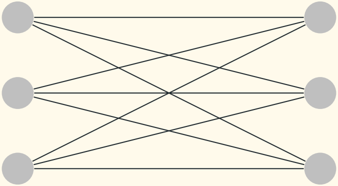
  - Can orbifold a $\mathbb{Z}_2$ subgroup, or the full $\mathbb{Z}_2 \times \mathbb{Z}_2$. E.x.

$$Z[\alpha_a, \alpha_b, \beta_a, \beta_b] \mapsto \frac{1}{2}\sum_{\delta} (-1)^{\delta_a\beta_b - \delta_b\beta_a} Z[\alpha_a, \alpha_b, \delta_a, \delta_b]\,. \qquad (18)$$

- **Brute force approach**. Can represent all of these operations as matrices acting on a basis of twisted partition functions.
  - The group of all topological manipulations is isomorphic to $T_B := O(2,2;\mathbb{F}_2)$. 72 elements.
  - The subgroup of all manipulations minus electric-magnetic type is $T_{B,0} := \mathbb{Z}_2 \rtimes GL(2;\mathbb{F}_2)$. 12 elements.

Bxamples: $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold Groupoid 2/3

- To present orbifold groupoid:
  - Form quotient $T_B/T_{B,0}$; a coset is collection of theories related by non-orbifold topological manipulations.
  - Connect two cosets by a line if I can apply orbifold operation to theory in one coset and obtain theory in the other coset.



- Rigorous study of holomorphic VOAs [Gaiotto, Johnson-Freyd '18]

BXAMPLES: $\mathbb{Z}_2 \times \mathbb{Z}_2$ ORBIFOLD GROUPOID 3/3

(2+1)d Perspective

- $A = \mathbb{Z}_2^2$. Collection of anyons in bulk is qm double $A \times \hat{A}$.
  - Fusion rule is inherited from group structure.
  - Topological spin of anyon labelled $(a, \hat{a})$ is $\chi_{\hat{a}}(a)$.
- Symmetries of this (2+1)d TFT are given by maps on anyons preserving the topological spin, so $T_B = O(A \oplus \hat{A}, \chi)$.
  - Mathematically, this group is the group of "braided auto-equivalences" of $Z(\mathrm{Vec}_A)$.
  - Equivalent to finding invertible interfaces from TFT to itself.
- The group of symmetries that fix the Dirichlet boundary conditions is $T_{B,0} = H^2(A, U(1)) \rtimes \mathrm{Aut}(A)$.
  - Mathematically, auto-equivalences of Vec_A.

Fermions

FERMIONS: (1+1)D FERMIONIC THEORIES

- By (1+1)d fermionic theory, mean QFT that requires **(s)pin structure** and has operators of half-integral spin.
  - Comes equipped with **Grassmann parity** $(-1)^F$ or $\mathbb{Z}_2^f$ which commutes with other symmetries

$$0 \to \mathbb{Z}_2^f \to G_f \to G \to 0\,. \tag{19}$$

- Background connections are "spin-G_f" connections, whose curvature is $w_2(M)$.
  - When G_f splits as $\mathbb{Z}_2^f \times G$ this is the same as independent choices of G connection and spin-structure.
- Tempting to assume if $G_f = \mathbb{Z}_2^f \times G$ that we can orbifold bosonic subgroups $H \leq G$ as before; roughly true.
  - In fermionic theories, we describe anomalies by the **supercohomology** $sH^3(G, U(1))$ and discrete torsion by $sH^2(G, U(1))$.
  - There are simply more invertible phases we can stack once fermions are involved. E.x. just $(-1)^F$ already has $(-1)^{\mathrm{Arf}[\eta]}$.

Fermions: GSO Projection

- We can also orbifold fermionic subgroups $H_f \leq G_f$, by making the spin-H_f connection dynamical.
  - This is a **GSO projection** or "GSO Orbifold".
  - Resulting theory may be bosonic or fermionic.

Example: $(-1)^F$ Theory

If we have a theory with $(-1)^F$ symmetry, we GSO project by summing over spin-structures.

$$Z_{T_b}[\alpha] = \frac{1}{\sqrt{|H^1(M, \mathbb{Z}_2)|}} \sum_{\eta} (-1)^{\mathrm{Arf}[\alpha+\eta]+\mathrm{Arf}[\eta]} Z_{T_f}[\eta]$$

Can also stack with $(-1)^{\mathrm{Arf}[\eta]}$ theory first to get a different GSO projection. Alternatively, there is one fermionic theory and two different GSO projections.

Fermions: sDW Gauge Theory

- Introduce **super Dijkgraaf-Witten** theory which has a dynamical bulk spin-G_f connection, with topological action $\mu_3 \in sH^3(G, U(1))$ denote it $\mathrm{sDW}[G_f]_{\mu_3}$.
  - This is a **bosonic** TFT described by some normal MTC.
  - Contrast to a "fermionic Dijkgraaf-Witten" theory which has Grassmann parity ungauged in bulk, hence is a spin-TFT.
- One difference is that topological spin of Wilson lines can be -1 now as opposed to $+1$ (depending on how $\mathbb{Z}_2^f$ on the corresponding G_f irrep).
- Dirichlet BCs require choice of boundary value for spin-G_f connection.
  - In the previous bosonic study this meant choosing a (1+1)d A background connection.
  - Now a Dirichlet BC involves choosing a (1+1)d spin-structure. Hence Dirichlet BCs for the sDW theories are **fermionic**.

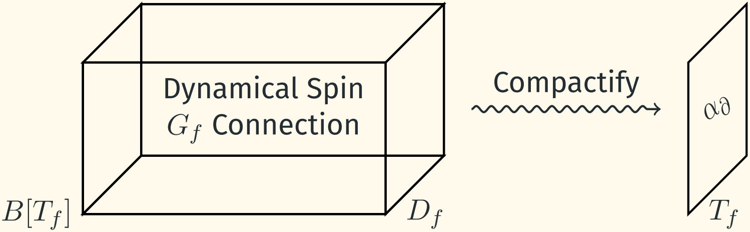
Fermions: Toric Code is a sDW Theory

Simplest sDW theory we can construct is the **pure spin structure gauge theory**.

- Consider the trivial (2+1)d spin-TFT. As a spin-TFT it necessarily requires a choice of spin structure.
- Sum over all spin structures in the trivial spin-TFT to produce the **bosonic shadow** [Bhardwaj, Gaiotto, Kapustin '17].
- This is *also* a gauge-theoretic realization of the toric code. Call it $\mathbb{Z}_2^f$ gauge theory.
  - Now its Wilson line is the f anyon.
  - Wilson line terminates on a topological fermionic B_f.
  - Two ways to identify disorder defects with e or m.

Fermions: Decoupling and GSO Projection

- Perform same construction as in the bosonic case, coupling (1+1)d fermionic theory to boundary of sDW theory.
  - (1+1)d theory T_f becomes a **bosonic** boundary condition $B[T_f]$
  - Recover T_f by placing **fermionic** topological Dirichlet BC on other end of slab.



- Get a **GSO projection** of T_f by placing *bosonic* Dirichlet boundary conditions on end of slab.
  - GSO projection is a way to produce a boundary condition for a $\mathbb{Z}_2$ gauge theory from a $\mathbb{Z}_2^f$ gauge theory.

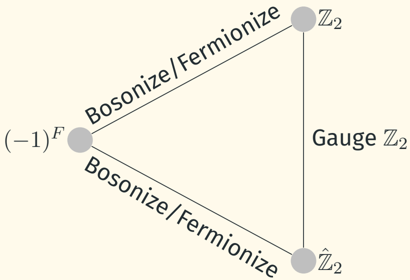
Fxamples

Fxamples: $(-1)^F$ Orbifold Groupoid

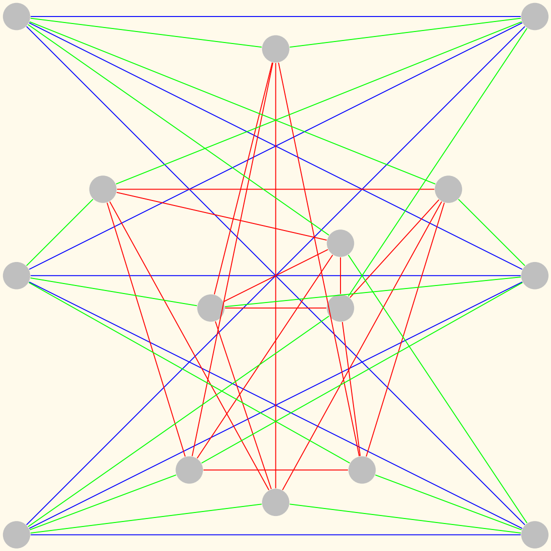
- Suppose a (1+1)d theory T_f with only $(-1)^F$ symmetry.
  - No non-trivial automorphisms.
  - One non-trivial invertible phase, $(-1)^{\mathrm{Arf}[\eta]}$

$$Z_f[\eta] \mapsto (-1)^{\mathrm{Arf}[\eta]} Z_f[\eta] \tag{20}$$

  - Can also GSO project.
- Orbifold groupoid is just



Fxamples: $\mathbb{Z}_2 \times \mathbb{Z}_2^f$ Orbifold Groupoid



Conclusion

Conclusion: Recap

- Orbifolds are discrete gauge theories. Orbifolding is just some topological manipulation of a theory.
  - Leaves local dynamics the same.
  - Changes partition function and correlation functions on manifolds of non-trivial topology.
- Given some theory in (1+1)d dimensions, you can use a (2+1)d TFT to decouple topological data from local data.
  - Topological data of the (1+1)d theory can be massaged by operations in the (2+1)d TFT, where technology is developed.
- Define and construct orbifold groupoids to understand what happens when we successively orbifold a theory.
  - This answers: how many new theories can we get?
- TFT argument runs parallel for fermions. GSO projections (and Jordan-Wigner) are orbifolds that come from compactifying on a segment with fermionic topological BCs.

Conclusion: Future Problems

- Flesh out examples for non-Abelian theories (full fusion categories of symmetries) and theories with fermions [Lou, Shen, Chen, Hung '20].
  - More generally, some full higher category of generalized symmetry defects. Something even larger than a fusion category can handle [Gaiotto, Johnson-Freyd '19].
- Non-orientable surfaces and time-reversal symmetry should be studied, should work similar in principle.
- There are some fun mathematical problems left.