Duality Defects in E_8

for PI Quantum Fields & Strings

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30/Apr/2021

Overview

1 Introduction

2 Two Important Fusion Categories

3 CFT and Partition Functions

4 The Ising CFT

5 The E_8 Theory

6 Conclusion

Introduction

Introduction: Why Defects?

- **Defects** are everywhere in physics.
  - Extended operators: lines, surfaces, ...
  - Understanding boundaries and interfaces of QFTs tells us about the theory itself. RG flows, dualities, ...
- In past few years, focusing on **Topological Defect Lines** modeled by **Fusion Categories** has been extremely profitable. Even just focusing on 2d we have
  - Modeling statistical systems with inhomogeneities [2008.08598]
  - Constraints for RG flows, generalizes 't Hooft anomaly matching, etc. [1802.04445], [2008.05960], [2008.07567]
  - Constraints for modular bootstrap [1904.04833], [2004.12557]
  - Clarifying fermionization and orbifold relationships between theories [1909.01425]
  - Constraints for quantum gravity [2006.10052]
- Before getting into that, we need a model organism...

Introduction: Symmetry Defects 1/2

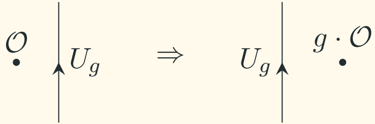
- **Symmetry Defects** are the model topological defect.
- Suppose we have a theory with (0-form) global symmetry group G.
  - For every $g \in G$, define $U_g(M^{(d-1)})$ by cutting spacetime along the codim. 1 manifold $M^{(d-1)}$ and inserting the appropriate g-action in the complete set of states associated with $M^{(d-1)}$.

$$U_g(M^{(d-1)})U_h(M^{(d-1)}) = U_{gh}(M^{(d-1)}) \tag{1}$$

  - Ward identities imply we may freely deform the supports of U_g but must be careful when moving past another charged operator.
- Symmetry defects are **invertible**.

Introduction: Symmetry Defects 2/2

- Thus, in 2d: *each element $g \in G$ gives an invertible topological defect line U_g*
  - They fuse naturally: $U_g \otimes U_h = U_{gh}$
  - They are invertible: $U_g \otimes U_{g^{-1}} = U_1$.
  - If it passes over a local operator insertion $\mathcal{O}$, it acts appropriately



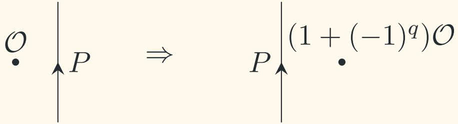
- From here out, we will focus on topological defect lines in 2d

Introduction: Non-Invertible Defects

- There are also **non-invertible defects**.
  - For example, if you have a $\mathbb{Z}_2$ symmetry, define the operator $P = U_0 \oplus U_1$ by

$$\langle \cdots (U_0 \oplus U_1)(\gamma) \cdots \rangle = \langle \cdots U_0(\gamma) \cdots \rangle + \langle \cdots U_1(\gamma) \cdots \rangle \quad (2)$$

  - Such an operator must be non-invertible because "it loses information." i.e. it projects out $\mathbb{Z}_2$ charged operators.



Goal for Today

Describe an interesting species of non-invertible TDLs in 2d CFTs known as **duality defects** or **Tambara-Yamagami lines**.

Introduction: The Plan

Question: *What is a duality defect and how do I find them?*

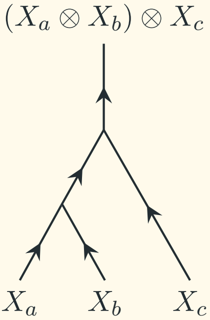
Plan:

1. Introduce two basic fusion categories by showing pictures of TDLs and what phenomena they capture.
2. Describe how topological defects act in 2d CFT, and look at a special class called Verlinde line defects.
3. Look at the Ising CFT and understand its duality defect line in detail. Try to give a physical picture of how it explains important properties of the Ising CFT.
4. Briefly introduce the chiral E_8 theory, and attempt to study its duality defects.
5. Explain what's left to do (if anything) with understanding duality defects in the chiral E_8 theory.

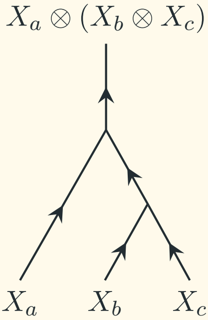
Two Important Fusion Categories

Fusion Categories: Generalities 1/2

- **Fusion categories** are categories where the objects are TDLs, subject to consistency axioms we'd expect for 2d lines.
  - e.g. if two lines come sufficiently close together with the same orientation they fuse
  - e.g. rules for multiple fusions and pentagon identity



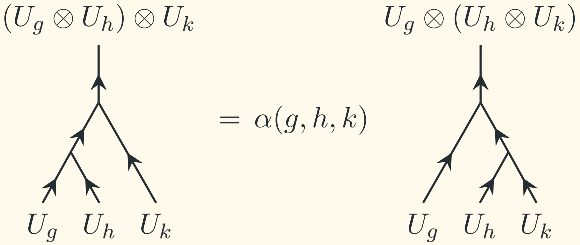
$$= \alpha(X_a, X_b, X_c)$$



- All-in-all, to talk about a fusion category, we need:
  1. The (finite) list of simple objects $\{a, b, c, \dots\}$
  2. How the simple objects fuse N^a_{bc}
  3. Associator data $\alpha(a, b, c)$ (or F-symbols)
- A **finite number of simple lines** is important!
  - As a non-example, consider a theory with $U(1)$ symmetry, then there is a continuous family of invertible symmetry lines labelled by θ, defined by the contour integral $e^{i\theta \int_\gamma ds^\mu j_\mu}$.
- This makes fusion categories more of a generalization of **finite groups** rather than arbitrary groups
  - This finiteness leads to "Ocneanu rigidity" which says you can't continuously deform fusion categories to others, and is why they put constraints on RG flows and generalize 't Hooft anomaly matching.

Fusion Categories: G-Graded Vector Spaces

- Given a **finite group** G, the topological defects associated to the group G form a fusion category in a natural way:
  1. Simple objects are elements of G.
  2. Fusion rules are just group composition $N_{g,h}^{a} = \delta_{a,gh}$
  3. The associator is given by some $\alpha \in H^3(G, U(1))$



  - This gives the fusion category mathematicians call Vec_G^{α}
  - In physics language, α is the anomaly for the G symmetry, the ambiguity in coupling to a background G connection.

- TY Categories are categorifications of the fusion algebra of an abelian group A **extended by one additional object** $\mathcal{N}$.
- The data are as follows:
  1. The simple objects are $A \cup \{\mathcal{N}\}$
  2. The fusion rules are the same as Vec_A but also

$$X_{\mathcal{N}} \otimes X_a = X_a \otimes X_{\mathcal{N}} = X_{\mathcal{N}}\,, \qquad X_{\mathcal{N}} \otimes X_{\mathcal{N}} = \bigoplus_{a \in A} X_a \tag{3}$$

  3. The (non-identity) associators are

$$\alpha_{a,\mathcal{N},b} = \chi(a,b)\mathbb{1}_{\mathcal{N}}\,, \tag{4}$$

$$\alpha_{\mathcal{N},a,\mathcal{N}} = \bigoplus_b \chi(a,b)\mathbb{1}_b\,, \tag{5}$$

$$\alpha_{\mathcal{N},\mathcal{N},\mathcal{N}} = (\tau\chi(a,b)^{-1}\mathbb{1}_{\mathcal{N}})_{a,b}\,. \tag{6}$$

- Mathematicians call this $\mathrm{TY}(A,\chi,\tau)$.

CFT and Partition Functions

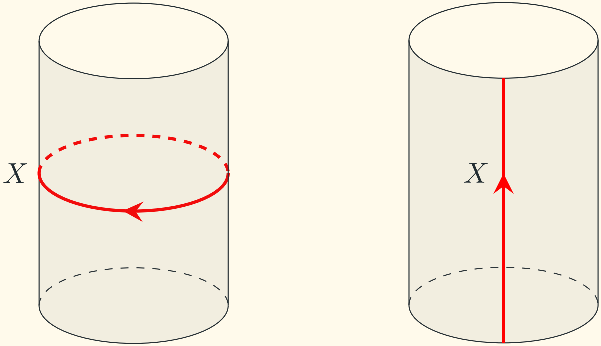
CFT: Defining a Defect Concretely (Plane)

- Since a defect line is an inhomogeneity in our system, we must specify boundary conditions for fields on either side of the defect line.
  - Focusing on 2d CFT, we describe how a defect line acts on states of the Hilbert space $\mathcal{H}$.
- In the plane, a defect line X defines an operator $\hat{X}$ on the Hilbert space of states on a circle $\mathcal{H}$ as follows:
  - Place $\phi \in \mathcal{H}$ at the origin
  - Place the defect X on the unit circle
  - This defines the state $\hat{X}\phi \in \mathcal{H}$
- X is topological if $\hat{X}$ commutes with the stress tensor

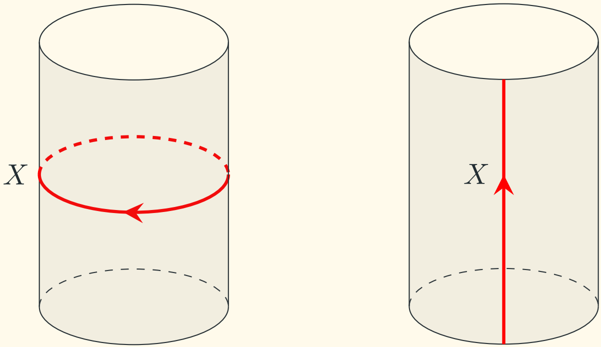
$$[\hat{L}_n, \hat{X}] = 0 = [\hat{\bar{L}}_n, \hat{X}] \tag{7}$$

CFT: Defining a Defect Concretely (Cylinder)

- Said more cylindrically, prepare ϕ in the infinite past and have X waiting for it half-way up the cylinder, then the out state is $\hat{X}\phi$.



- The Hilbert space $\mathcal{H}_X$ of operators upon which X can end is the Hilbert space of the theory on a circle, except with a single future oriented X defect piercing the circle.



- Partition function with X-twist in Euclidean time

$$Z^X(\tau,\bar\tau) = \mathrm{Tr}_{\mathcal{H}}\, \hat{X}\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\bar{L}_0-\frac{c}{24}}\ , \tag{8}$$

- Partition function with X-twist in space

$$Z_X(\tau,\bar\tau) = \mathrm{Tr}_{\mathcal{H}_X}\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\bar{L}_0-\frac{c}{24}}\ . \tag{9}$$

CFT: Verlinde Line Defects

- If we are working with a diagonal RCFT, we can **define** a class of TDLs, called **Verlinde Lines**, which commute with the entire chiral algebra $\mathcal{A}$
  - In one-to-one correspondence with primaries
  - Act as

$$\hat{X}_i \left|\phi_j\right\rangle = \frac{S_{ij}}{S_{0j}} \left|\phi_j\right\rangle \, , \tag{10}$$

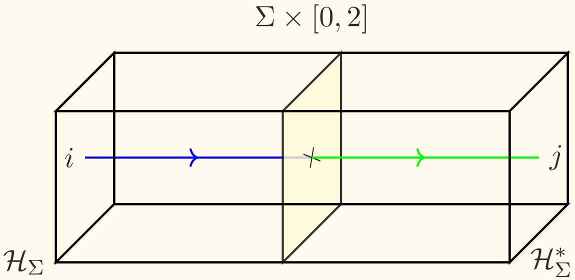
  - Fuse as

$$X_i \otimes X_j = \bigoplus_k N_{ij}^k X_k \, . \tag{11}$$

- Natural consequence of the bulk-boundary relationship between a 2d RCFT and its associated 3d TFT
  - Topological defects coming from anyons of bulk 3d TFT, brought to boundary where 2d RCFT data lives

CFT: Verlinde Lines and TFT

- Consider a WZW model on Σ. This is equivalent to 3d CS on an interval $\Sigma \times [0,2]$, with chiral and anti-chiral boundary conditions on the ends.
  - For non-trivial modular invariant, we insert a non-trivial surface operator that "glues" the information from chiral algebra $\mathcal{A}$ to local piece from anti-chiral algebra $\bar{\mathcal{A}}$.
  - Explains why Verlinde lines have same fusion rule as operators



The Ising CFT

Ising: The Setup

- Consider the 2d $c = \frac{1}{2}$ Ising model. It has 3 Virasoro primaries

$$\mathbb{1}_{0,0}, \quad \epsilon_{\frac{1}{2},\frac{1}{2}}, \quad \sigma_{\frac{1}{16},\frac{1}{16}} \tag{12}$$

  - The fusion rules are

$$[\sigma][\sigma] = [\mathbb{1}] \oplus [\epsilon], \quad [\sigma][\epsilon] = [\sigma], \quad [\epsilon][\epsilon] = [\mathbb{1}]. \tag{13}$$

- This also means it has 3 Verlinde lines, $X_{\mathbb{1}}$, X_ϵ, and X_σ.
  - These turn out to be all the TDLs of the Ising
  - They act on states by

	$\mathbb{1}_{0,0}$	$\epsilon_{\frac{1}{2},\frac{1}{2}}$	$\sigma_{\frac{1}{16},\frac{1}{16}}$
$X_{\mathbb{1}}$	1	1	1
X_ϵ	1	1	-1
X_σ	$\sqrt{2}$	$-\sqrt{2}$	0

Ising: It has $\mathbb{Z}_2$ Symmetry

- The Ising model has a $\mathbb{Z}_2$ symmetry. Clearly $X_{\mathbb{1}}$ is the identity, and X_ϵ is the $\mathbb{Z}_2$ symmetry line. X_ϵ acts non-trivially on the $\mathbb{Z}_2$ charged operators ($\sigma_{\frac{1}{16},\frac{1}{16}}$, $\psi_{\frac{1}{2},0}$, $\bar{\psi}_{0,\frac{1}{2}}$, $\ldots$)
  - We have the usual Hilbert space of local operators $\mathcal{H}$. Think of as the endpoint operators for the trivial line $X_{\mathbb{1}}$.

$$\mathcal{H}_{\mathbb{1}} = \mathrm{span}\{\mathbb{1}_{0,0}, \epsilon_{\frac{1}{2},\frac{1}{2}}, \sigma_{\frac{1}{16},\frac{1}{16}}\}\,. \tag{14}$$

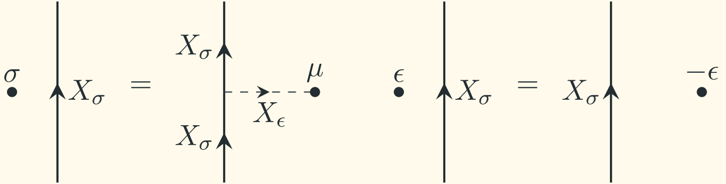
  - We also have the Hilbert space of **twist operators** which have to have an X_ϵ topological tail.

$$\mathcal{H}_\epsilon = \mathrm{span}\{\mu_{\frac{1}{16},\frac{1}{16}}, \psi_{\frac{1}{2},0}, \bar{\psi}_{0,\frac{1}{2}}\}\,. \tag{15}$$

- In the statistical model, the $\mathbb{Z}_2$ symmetry line X_ϵ is called a “line of frustration,” and μ is called the “disorder operator.”
  - Famously, in the (not necessarily critical) Ising model, $\langle\sigma(z_1)\cdots\sigma(z_n)\rangle_\beta = \langle\mu(z_1)\cdots\mu(z_n)\rangle_{\tilde{\beta}}$. **But why?**

ISING: WHAT IS X_σ?

- A statistical interpretation of the X_σ line is not immediately clear, but can be realized by studying how it acts on the primaries in the CFT picture.
- Sweeping X_σ past a σ insertion, we are left with a μ insertion and a topological X_ϵ tail.
- Sweeping X_σ past an ϵ insertion, it leaves behind $-\epsilon$.



ISING: X_σ IS A DUALITY DEFECT!

- X_σ is a **duality defect!**
  - Perhaps we noticed that from the fusion rules

$$[\sigma][\sigma] = [\mathbb{1}] \oplus [\epsilon]\,, \quad [\sigma][\epsilon] = [\sigma]\,, \quad [\epsilon][\epsilon] = [\mathbb{1}]\,. \tag{16}$$

- First note that it relates correlators of order operators to disorder operators at the critical point.
  - For example, on the sphere

  - So $\langle \sigma(z_1) \cdots \sigma(z_n) \rangle_{\beta_c} = \langle \mu(z_1) \cdots \mu(z_n) \rangle_{\beta_c}$

- Similarly, on the torus
  - The KW defect separates the Ising from its orbifold.
- The action on ϵ also explains why Kramers-Wannier duality is **true for all temperatures.**
  - ϵ is relevant in the Ising CFT, flows the Ising to high temperature or low temperature phases.
  - Since X_σ changes the sign of ϵ then relations hold like

$$\left\langle \sigma(x)\sigma(x')e^{-\lambda\int \epsilon(y)d^2y} \right\rangle = \left\langle \mu(x)\mu(x')e^{+\lambda\int \epsilon(y)d^2y} \right\rangle \tag{17}$$

- Suppose I want to know the **defected partition function**

$$Z^{X_\sigma}(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, \hat{X}_\sigma q^{L_0-\frac{1}{48}}\bar{q}^{\bar{L}_0-\frac{1}{48}}\,, \tag{18}$$

$$= \sum_i \frac{S_{i\sigma}}{S_{0\sigma}}\chi_i(\tau)\bar{\chi}_i(\bar{\tau})\,, \tag{19}$$

$$= \sqrt{2}|\chi_0|^2 - \sqrt{2}|\chi_{\frac{1}{2}}|^2\,. \tag{20}$$

- Could I have guessed this without knowing about Verlinde lines?
  - No. But I can get close using $X_\sigma \otimes X_\sigma = X_{\mathbb{1}} \oplus X_\epsilon$.

Recap and Reorientation

Recap:

1. Defects are important, they can be topological, invertible, non-invertible, and so-on.
2. Fusion categories exist. Symmetry defects behave like Vec_G^α, and symmetry defects plus a "duality defect" behave like $\mathrm{TY}(A, \chi, \tau)$.
3. The Ising model has a duality defect, which can be obtained by looking at it's Verlinde lines, and it describes an isomorphism from $\mathrm{Ising} \cong \mathrm{Ising}/\!/\,\mathbb{Z}_2$.

Reorientation:

- Recall our question: *What is a duality defect and how do I find them?*
- We now know that a duality defect is something that relates a CFT to its orbifold, but we relied heavily on X_σ being a Verlinde line.

The E_8 Theory

E_8 Theory: It Exists and is Unique

- To have a full CFT, we combine a chiral algebra $\mathcal{A}$ and anti-chiral algebra $\bar{\mathcal{A}}$ in some modular invariant way.
  - This requires, at minimum, that $c_L - c_R \in 24\,\mathbb{Z}$.
  - Usually (but not always) we assume $\mathcal{A} = \bar{\mathcal{A}}$ and this gives us $c_L - c_R = 0$.
- If we allow ourselves to study things whose modular non-invariance can be fixed with a (2+1)d gravitational Chern-Simons term, then its not outrageous to look at CFTs with $c_L - c_R \in 8\,\mathbb{Z}$.
- It turns out there is a unique holomorphic CFT with $c_L = 8$ (and $c_R = 0$ by defn.) which I will call the **chiral E_8 theory**.
  - Since the theory is unique, it should have duality defects just like the Ising, for every non-anomalous symmetry.
  - Since the theory is holomorphic, it has no Verlinde lines! i.e $\mathrm{Rep}(V_{E_8})$ is trivial. So we give up on finding duality defects...

E_8 Theory: What is it really though?

- You can think of the chiral E_8 theory as being just the chiral algebra/VOA for the $(E_8)_1$ WZW model.
- A different, equivalent, characterization is as the E_8 LVOA.
  1. Start with the 8 dimensional E_8 lattice L_{E_8}.
  2. For **every vector** $\alpha \in L_{E_8}$, you have a state $|\alpha\rangle$ created by the **vertex operator** Γ_α. By definition $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$. Also
  $$\Gamma_\alpha \Gamma_\beta = \epsilon(\alpha,\beta)\Gamma_{\alpha+\beta} \tag{21}$$
  3. For **each root** we get a free **independent boson**, whose modes satisfy usual Heisenberg commutation relation
  $$[a^i_m, a^j_n] = m\delta^{ij}\delta_{m+n,0}\,. \tag{22}$$
  4. They satisfy
  $$a^i_n\,|\alpha\rangle = 0\,, \qquad \text{if } n > 0,, \tag{23}$$
  $$a^i_0\,|\alpha\rangle = \alpha^i\,|\alpha\rangle\ . \tag{24}$$

E_8 Theory: Partition Function

- Working with all this, we can compute the partition function

$$Z_{E_8}(\tau) = \frac{1}{2\eta(\tau)^8}\left(\theta_1(\tau)^8 + \theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8\right) \qquad (25)$$

- This is just the partition function for (the GSO projection of) a theory of **16 chiral fermions**.
- Moreover, the Ising CFT is just (the GSO projection of) a free (non-chiral) fermion

$$Z_{\text{Ising}}(\tau,\bar{\tau}) = \frac{1}{2|\eta(\tau)|}\left(|\theta_1(\tau)| + |\theta_2(\tau)| + |\theta_3(\tau)| + |\theta_4(\tau)|\right) \qquad (26)$$

Secret Fermion Interlude: Dual Symmetry

- By now, people know if you have a fermionic theory T_f with $(-1)^F$ symmetry (and $c_L - c_R \in 8\,\mathbb{Z}$), then you can gauge the diagonal spin-structure (in two different ways) to produce bosonic theories with a non-anomalous $\mathbb{Z}_2$ symmetry.
  - The $(-1)^F$ symmetry defect in the fermionic theory becomes a the $\mathbb{Z}_2$ symmetry defect in the bosonic theory

- What happens to the chiral fermion parity $(-1)^{F_L}$
  - A twist by the **chiral fermion parity** becomes a **duality defect** $\mathcal{N}$ in the bosonic theory!

- Explicitly, if we do the GSO projection of the free fermion CFT, but insert a $(-1)^{F_L}$ symmetry defect before GSO projecting

$$\frac{1}{2}\sum_{\rho} Z_{\mathrm{Maj.}}[\rho_1,\rho_2+1]\bar{Z}_{\mathrm{Maj.}}[\rho_1,\rho_2] \propto \sqrt{2}|\chi_0|^2 - \sqrt{2}|\chi_{\frac{1}{2}}|^2 \,. \qquad (27)$$

- More generally, this comes from the fact that $(-1)^{F_L}$ has one unit of the "mod 8 anomaly" coming from

$$\mathrm{Hom}(\Omega_3^{\mathrm{Spin}}(B\,\mathbb{Z}_2), U(1)) = \mathbb{Z}_8 \qquad (28)$$

  - For n Majorana fermions the bosonization is the $\mathrm{Spin}(n)_1$ WZW model
  - If $n = 1, 7 \pmod 8$ you get $\mathrm{TY}(\mathbb{Z}_2, 1, +1/\sqrt{2})$
  - If $n = 3, 5 \pmod 8$ you get $\mathrm{TY}(\mathbb{Z}_2, 1, -1/\sqrt{2})$

E_8 Theory: $\mathbb{Z}_2$-TY Defects in E_8

- The chiral E_8 theory is just 16 Majorana-Weyl fermions. So we should get 4 different duality defects based on if we view it as coming from the GSO projection of

$$Z^1_{\text{Maj.}}[\rho]Z^{15}_{\text{Maj.}}[\rho]\,, \quad Z^3_{\text{Maj.}}[\rho]Z^{13}_{\text{Maj.}}[\rho]\,, \tag{29}$$

$$Z^5_{\text{Maj.}}[\rho]Z^{11}_{\text{Maj.}}[\rho]\,, \quad Z^7_{\text{Maj.}}[\rho]Z^{9}_{\text{Maj.}}[\rho]\,. \tag{30}$$

- In which case **there are 4 $\mathbb{Z}_2$ duality defects in E_8**

$$Z_V[0, X_p] \propto \frac{1}{2}\sum_{\rho} Z^p_{\text{Maj.}}[\rho_1, \rho_2+1]Z^{16-p}_{\text{Maj.}}[\rho_1, \rho_2]\,, \tag{31}$$

$$= \frac{(\theta_3(\tau)\theta_4(\tau))^{\frac{p}{2}}}{2\eta(\tau)^8}\left(\theta_3(\tau)^{8-p} + \theta_4(\tau)^{8-p}\right)\,. \tag{32}$$

Conclusion

Conclusion: Higher Order Symmetries 1/2

- What about higher order symmetries? i.e. $\mathbb{Z}_{n>2}$.
  - Maybe we can parafermionize and ask about... chiral parafermion number? This seems too hard.
- The clue to a general answer comes from when we were trying to guess the defected partition function for the Ising.
  - Think about operators of the Ising CFT relevant in $\mathbb{Z}_2$ orbifold
- The sign difference for the $\epsilon_{\frac{1}{2},\frac{1}{2}}$ primary roughly came from "the Ising and its orbifold see ϵ differently."

Conjecture: $\mathbb{Z}_m$ TY defects of $V = V_{E_8}$ are obtained as follows...

1. Find all non-anomalous (therefore gaugeable) $\mathbb{Z}_m$ symmetries of V. The uncharged operators form a sub-VOA

$$V^{\mathbb{Z}_m} \subset V \tag{33}$$

2. $\mathbb{Z}_m$-twisted Hilbert spaces of V decompose over $\mathrm{Irr}(V^{\mathbb{Z}_m})$.

	$[V/\mathbb{Z}_m]$	$[V/\mathbb{Z}_m]^1$	$\cdots$	$[V/\mathbb{Z}_m]^{m-1}$
V	$\mathcal{H}_0^0$	$\mathcal{H}_0^1$	$\cdots$	$\mathcal{H}_0^{m-1}$
V_1	$\mathcal{H}_1^0$	$\mathcal{H}_1^1$	$\cdots$	$\mathcal{H}_1^{m-1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_{m-1}	$\mathcal{H}_{m-1}^0$	$\mathcal{H}_{m-1}^1$	$\cdots$	$\mathcal{H}_{m-1}^{m-1}$

3. Find order two automorphisms of $V^{\mathbb{Z}_m}$ which "swap the axes" corresponding to V and its orbifold in the table.

Conclusion: What We're Doing Now

- In terms of partition functions, the claim is that duality defected partition functions can be computed as

$$Z_V[0, \mathcal{N}] = \mathrm{tr}_{V^{\mathbb{Z}_m}}(\hat{\sigma} q^{L_0 - \frac{c}{24}}) \tag{34}$$

- We've tested this for $\mathbb{Z}_2$ duality defects in the chiral E_8 theory. We are working on testing this explicitly for the $\mathbb{Z}_3$ duality defects.
  - Obstruction: this involves computing twisted theta functions for lattices like E_6 and F_4 and reducing it to sums/products of regular theta functions. This is hard!

Conclusion: Recap

Recap:

1. Defects are important, they can be topological, invertible, non-invertible, and so-on.
2. Well behaved finite collections of TDLs are captured by fusion categories. They put constraints on RG and explain dualities.
3. The Ising model has a duality defect, which describes an isomorphism from $\mathrm{Ising} \cong \mathrm{Ising}//\mathbb{Z}_2$. It can be obtained just by looking at Verlinde lines.
4. The chiral E_8 theory should have duality defects too. You can't use Verlinde lines, but you can use fermionization to get the $\mathbb{Z}_2$ defects.
5. For higher defects, a new trick (and a proof it gives the desired result) will be needed.

To be continued...