Topological Aspects of QFT

for PSI Grad Student Talk Day

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Overview

Introduction: Outline and Goal of Talk

- Give a survey of: path integrals; symmetry operators; examples of topological data in QFT; and toric code.
- Goals:

1. Describe vaguely what a path integral is.
2. Show how symmetries correspond to invertible codimension 1 operators with topological support.

Bonus. Illustrate how we can blow up a (1+1)d theory to a (2+1)d gauge theory sandwich.

Key Words:

Axiomatic QFT; non-Lagrangian theories; topological field theory; anomalies; higher-form symmetries; categorical symmetries; SPT phases; cobordism; toric code; topological order; Chern-Simons theory; electric-magnetic duality; boundary conditions of gauge theory; orbifold

Path Integral

Path Integral: Basics

- To describe a d-dim **Quantum Field Theory** (QFT), we might start with the following data
  - A d-dim manifold M_d with a **space of fields** collectively denoted Φ. e.g. fields can be $\mathbb{R}$-val'd scalars; gauge fields; p-forms; or maps to a "target space" $\Phi : M_d \to W$.
  - A **local action functional** $S[\Phi]$, which is the action for the classical field theory we are quantizing.
- Objects of interest are **path integrals**, expressions like

$$\int D\Phi \, e^{-S[\Phi]} \tag{1}$$

  - If M_d closed, integral above gives **partition function** $Z[M_d]$
  - If M_d has boundary M_{d-1}, then integral depends on boundary conditions (BCs) for fields. e.g. if $\Phi|_{M_{d-1}} = \varphi$, then we can consider functions

$$\Psi[\varphi] = \int_{\Phi|_{M_{d-1}}=\varphi} D\Phi \, e^{-S[\Phi]} \in \mathcal{H}(M_{d-1}) \tag{2}$$

- If M_d has incoming and outgoing boundaries M_{d-1}^- and M_{d-1}^+ then the path-integral assigns a transition amplitude between $\langle \varphi^- |$ and $| \varphi^+ \rangle$.
- Taking these ideas to simplest and most axiomatic form gives e.g. **Atiyah-Segal axioms** for **topological field theory**.

A d dim topological field theory is a symmetric monoidal functor

$$Z : \mathrm{Bord}_{\langle n-1,n \rangle} \to \mathrm{Vec}_{\mathbb{C}} \,.$$

- **Correlation functions** in the theory are given by

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \int_{M_d} D\Phi \; \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \, e^{-S[\Phi]} \qquad (3)$$

SYMMETRIES

Symmetries: Noether's Theorem

- **Noether's theorem** associates a current to every continuous symmetry, which is conserved (on-shell).
- For simplicity, consider a classical theory of a real scalar ϕ. If $\delta L = 0$ under

$$\phi \mapsto \phi' = \phi + \epsilon\delta\phi\,, \tag{4}$$

then on-shell

$$J^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \quad \text{satisfies} \quad \partial_\mu J^\mu = 0\,. \tag{5}$$

- There is an associated **conserved charge** supported on a codimension 1 manifold $M_{d-1} \subset M_d$

$$Q(M_{d-1}) := \int_{M_{d-1}} \star J \tag{6}$$

Symmetries: Ward Identities 1/2

- Consider the same transformation in the path-integral for a correlation function (write $X := \phi(x_1)\cdots\phi(x_n)$)
  - From the point of view of the path integral, $\phi \mapsto \phi' = \phi + \epsilon\delta\phi$ is just a change of variables, so leaves it unchanged:

$$\langle X \rangle = \int D\phi \, X e^{-S[\phi]} \,, \tag{7}$$

$$= \int D\phi' \, X' e^{-S[\phi']} \,, \tag{8}$$

$$= \int D\phi' \, (X + \delta X) e^{-\left(S[\phi] - \int d^d x \, \partial_\mu J^\mu \epsilon(x)\right)} \,. \tag{9}$$

- If we assume $D\phi = D\phi'$ and expand to first order in ϵ, we get

$$\langle \delta X \rangle|_{\mathcal{O}(\epsilon)} = \int d^d x \; \langle \partial_\mu J^\mu X \rangle \, \epsilon(x) \tag{10}$$

Symmetries: Ward Identities 2/2

- Let's look at the LHS of this equation

$$\langle \delta X \rangle|_{\mathcal{O}(\epsilon)} = \int d^d x \; \langle \partial_\mu J^\mu X \rangle \, \epsilon(x) \tag{11}$$

  - We can write

$$\delta X|_{\mathcal{O}(\epsilon)} = \epsilon \sum_{i=1}^{n} \phi(x_1) \cdots \delta\phi(x_i) \cdots \phi(x_n) \, , \tag{12}$$

$$= \int d^d x \, \epsilon(x) \sum_{i=1}^{n} \delta(x - x_i) \phi(x_1) \cdots \delta\phi(x_i) \cdots \phi(x_n) \, .$$

Ward Identity

$$\langle \partial_\mu j^\mu(x) \phi(x_1) \cdots \phi(x_n) \rangle = \sum_{i=1}^{n} \delta(x - x_i) \phi(x_1) \cdots \delta\phi(x_i) \cdots \phi(x_n)$$

Topological Data

Topological Data: Symmetry Operators 1/2

- Recall the Noether charge defined for some $M_{d-1} \subset M_d$ as

$$Q(M_{d-1}) := \int_{M_{d-1}} \star J \tag{13}$$

- Ward identities imply that the support M_{d-1} is only **topologically** relevant in correlation functions

- We can exponentiate $Q(M_{d-1})$ to get a symmetry operator associated with a “less infinitesimal” symmetry.

$$U_g(M_{d-1}) \sim \exp(i\omega_a Q^a(M_{d-1}))\,. \tag{14}$$

Topological Data: Symmetry Operators 2/2

- More generally, for every $g \in G$, define $U_g(M_{d-1})$ by cutting spacetime along M_{d-1} and acting with appropriate g-action on the complete set of states associated with M_{d-1}.
  - Works for discrete G.
- Operations then compose like

$$U_g(M_{d-1})U_h(M_{d-1}) = U_{gh}(M_{d-1}) \tag{15}$$

Symmetry Defects

Symmetry defects are invertible topological codim 1 defects

Topological Data: Background Flat Connections

- What does a connection A do in a gauge theory? Tells us about about things like parallel transport. Holonomies.

- Our new view of symmetry operators as codimension 1 defects gives us a way to understand **background flat connections**.
  - Every background flat connection can be understood as a **network of defects**, labelling holonomy around non-contractible loops.
  - Gauge transformations just shift the domain walls around.
  - Every connection for a discrete gauge theory is flat.

Topological Data: SPT Phases

- Given a theory T with partition function Z_T on M_d and symmetry group G, we can consider the theory in the presence of a background G gauge field $Z[M_d; A]$.
- When G is discrete and abelian, we call this the **twisted partition function**. Connections are $A \in H^1(M_d, G)$ labelling holonomy around non-contractible cycles.
- A G **SPT Phase** is a gapped phase which is indistinguishable from trivial if one disregards symmetry.
  - In presence of background G gauge field, the partition function is a $U(1)$ number.
  - Actions classified by group cohomology or cobordism theory.
  - Can "stack" these SPT phases with more typical theories to change the topological properties of the theory.

Topological Data: Wilson and 't Hooft Lines

- G gauge theory has an interesting set of topological operators, called **Wilson Lines**; the worldline of an infinitely massive electrically charged particle

$$W_R(\gamma) = \mathrm{Tr}_R \, P \exp\left(i \int_\gamma A\right) \, . \tag{16}$$

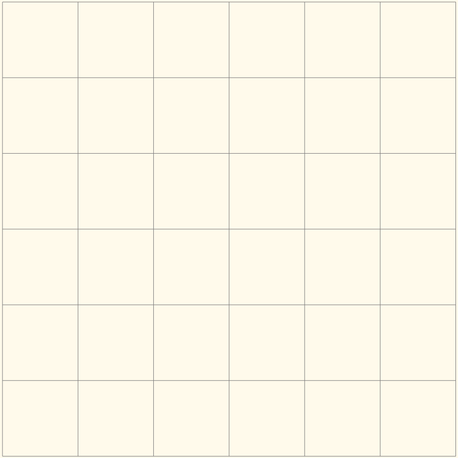
  - Labelled by a rep R of G.
- There are also **'t Hooft Lines**, which correspond to worldline of infinitely massive magnetically charged particle
  - Labelled by a cocharacter $\hat{\mu} : U(1) \to G$.
- Also dyonic lines with both electric and magnetic charge.
- Lines can braid around each other and collect phases, and fuse into different dyons.

Example: Toric Code

Example: Toric Code - General Facts

- **Toric Code** or (2+1)d $\mathbb{Z}_2$ **Gauge Theory** can be thought of in a number of different ways
  - As $\mathbb{Z}_2$ lattice gauge theory: start with 2d square lattice with spins on vertices; introduce "connection" on edges; gauge.
  - Simplest example of $\mathbb{Z}_2$ topological order
  - As $U(1)^2$ Chern-Simons at level $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$
  - As (2+1)d $U(1)$ gauge theory broken down to $\mathbb{Z}_2$
  - As "gauged $\mathbb{Z}_2$ SPT theory" or $\mathbb{Z}_2$ Dijkgraaf-Witten theory.
- We pre-highlight some key facts:
  1. It has 4 **lines** (simple)
  2. It has 2 **gapped boundary conditions** (bosonic)

Example: Toric Code - Lattice 1/2



- 2d square lattice with **spin-1/2** DoF on each **edge**.
- Introduce operators

$$A_v = \prod_{i \in v} \sigma_i^x \, , \quad B_p = \prod_{i \in p} \sigma_i^z \, .$$

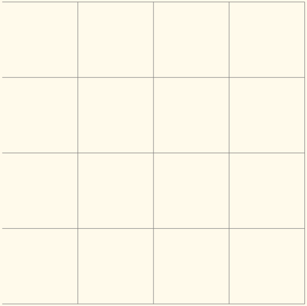
- The Hamiltonian is

$$H = -J \sum_v A_v - J \sum_p B_p$$

- The GS of Hamiltonian is

$$A_v \left|\psi\right\rangle = B_p \left|\psi\right\rangle = \left|\psi\right\rangle$$

Example: Toric Code - Lattice 2/2



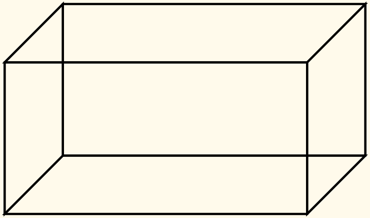
- Two types of **gapped boundary conditions**: rough and smooth.
- When e line approaches **rough boundary**, it can disappear!
- m line gets stuck at rough boundary! Boundary excitations for rough boundary are $\{1, m\}$

- This also shows off **electric-magnetic duality**: on the dual lattice we have the same physics, but e and m lines switch roles. The rough and smooth boundaries switch roles too.

Example: Toric Code - Continuum 1/2

$$M_3 = M_2 \times [0,1]$$

Dynamical $\mathbb{Z}_2$ Connection



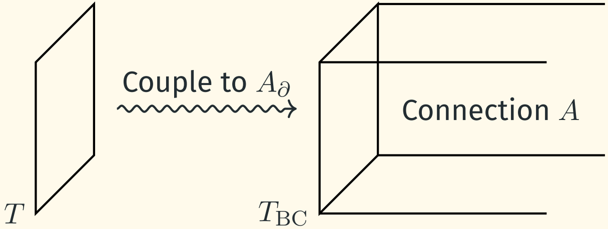
- If e line is the **Wilson Line**, then we saw there is a boundary (rough) where it can end. This is **Dirichlet BC** for bulk gauge field.
  - In this case, m is **'t Hooft line** and is a boundary excitation.
  - The smooth boundary looks like a **Neumann BC** for the bulk gauge field.

Example: Toric Code - Continuum 2/2

- Write $|D[A_\partial]\rangle$ for the **Dirichlet BC** that sets the 3d bulk connection to look like A_∂ at the boundary.
- It """"makes sense"""" that a **Neumann BC** (free BC) is like a sum over all Dirichlet BCs

$$|N\rangle := \sum_{B_\partial \in H^1(M_2, \mathbb{Z}_2)} |D[B_\partial]\rangle \tag{17}$$

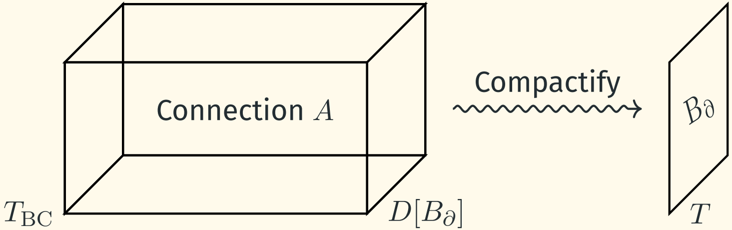
- Take some (1+1)d theory T with $\mathbb{Z}_2$ symmetry. Make a new enriched BC for the toric code by using A_∂ as background gauge field for T.



Example: Toric Code - 2d Dualities & 3d Bulk 1/2

- The state of the (2+1)d TFT given by the T-enriched BC is

$$|T\rangle := \sum_{A_\partial} Z_T[A_\partial]\,|D[A_\partial]\rangle \tag{18}$$



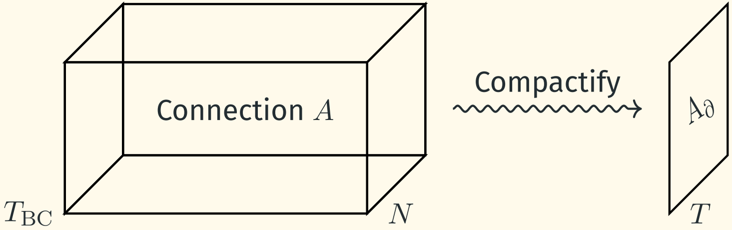
- Equip one end of a slab with T-enriched BC (i.e. $\langle T|$) and the other with **Dirichlet BC** $|D[\beta_\partial]\rangle$, then

$$\langle T|D[\beta_\partial]\rangle = \sum_{A_\partial} Z_T[A_\partial]\,\langle D[A_\partial]|D[\beta_\partial]\rangle = Z_T[B_\partial] \tag{19}$$

Example: Toric Code - 2d Dualities & 3d Bulk 2/2

- Use **Neumann BC** instead

$$|T\rangle := \sum_{A_\partial} Z_T[A_\partial]\,|D[A_\partial]\rangle \tag{20}$$



- Then we get the following (1+1)d gauged partition function from compactification instead:

$$\langle T|N\rangle \propto \sum_{A_\partial,\beta_\partial} Z_T[A_\partial]\,\langle D[A_\partial]|D[\beta_\partial]\rangle = \sum_{A_\partial} Z_T[A_\partial] \tag{21}$$

Conclusion

Recap and Outlook

Take away slogans from today:

1. The path integral associates a number to a closed manifold and a space of states to a boundary (or interface).
2. Symmetries correspond to invertible codimension 1 operators with topological support.
3. Boundary conditions for gauge theories have a rich structure and can be used to probe the gauge theory or other theories.

Fin