The $3d$ $O(N)$ Model on the Celestial Circle

for Celestial Holography '22 (Gong Show)

Justin Kulp

Work in Progress with
Diego García-Sepúlveda,
Alfredo Guevara, and
Jingxiang Wu

04/Feb/2022

Motivation

- What can Celestial Holography teach us about the structure of Quantum Field Theory?
    1. The **optical theorem** applies to the imaginary piece of the 4-point function. How does unitarity extend to the full correlation function?
    2. How does the **CCFT spectrum** emerge from the principal continuous series? What does it mean for scattering amplitudes?
    3. Which conclusions hold **beyond tree level**?
- Last point is particularly pressing because celestial amplitudes are highly sensitive to UV physics.
- The $O(N)$ **model** shines light on these questions, even in a strongly-coupled regime.

The $O(N)$ Model in $3d$

- The $O(N)$ **model** is a theory of N real scalars with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a - \frac{1}{2}\mu_0^2\varphi^a\varphi^a - \frac{\lambda_0}{8N}\left(\varphi^a\varphi^a\right)^2 \tag{1}$$

  - Canonical example of **spontaneous symmetry breaking**. Leaving $N-1$ massless pions π^a, and massive σ boson
  - Study perturbatively in λ or in $1/N$ **expansion** with λ fixed
- To solve in $1/N$, introduce Hubbard-Stratonovich field χ and compute Γ_{eff} [Coleman Jackiw Politzer], [Gross Neveu].
  - Pion propagator is $D_{\pi^a\pi^b} = \delta^{ab}/p^2$.
  - Mixing between σ and χ. Propagators can be computed

$$D_{\sigma\sigma} = \frac{p+\frac{\lambda}{16}}{p\left(p^2+\frac{\lambda}{16}p-2\mu^2\right)}\,, \quad D_{\chi\chi} = -\frac{\lambda}{N}\frac{p^2}{p^2+\frac{\lambda}{16}p-2\mu^2} \tag{2}$$

- Both $D_{\sigma\sigma}$ and $D_{\chi\chi}$ have **poles on second sheet** at

$$p_\pm = -\frac{\lambda}{32} \pm i\sqrt{-2\mu^2-\left(\frac{\lambda}{32}\right)^2} \tag{3}$$

- Can compute **3 pt amplitude**: $\pi^a \pi^b \to \sigma$

$$T_3^{ab}(p_\pm) = \sqrt{\frac{-\mu^2 \lambda}{N}} \frac{-2p_\pm}{\sqrt{p_\mp(p_\mp - p_\pm)}} \delta^{ab} \tag{4}$$

  - 2 resonances give 2 CCFT 3pt functions $C(\Delta_\pi, \Delta_\pi; \Delta_{\sigma_\pm})$
- **4pt amplitude**: $\pi^a \pi^b \to \pi^c \pi^d$ to leading order in $1/N$

$$T_4^{ab,cd}\left(\omega^2, -\frac{\omega^2}{z}\right) = -\frac{\lambda}{N} \frac{\omega^2 \delta^{ab} \delta^{cd}}{\omega^2 + i\frac{\lambda}{16}\omega + 2\mu^2} + (t\text{-diag}) + (u\text{-diag})\,. \tag{5}$$

- Mellin transform to produce CCFT 4pt function
  - Mellin integral free of poles on real axis and **convergence strip** includes 1d **principal continuous series** $\Delta_\pi \in \frac{1}{2} + i\mathbb{R}$

The CCFT Four Point Function

CCFT Four Point Function

In the physical s-channel $z \in (1, \infty)$

$$f_s^{abcd}(z) = \mathcal{N}\mathcal{N}_\alpha \frac{z}{\sqrt{z-1}} \left(\delta^{ab}\delta^{cd} e^{i\pi\alpha} + \delta^{ac}\delta^{bd} z^\alpha + \delta^{ad}\delta^{bc} \left(\frac{z}{z-1}\right)^\alpha \right)$$

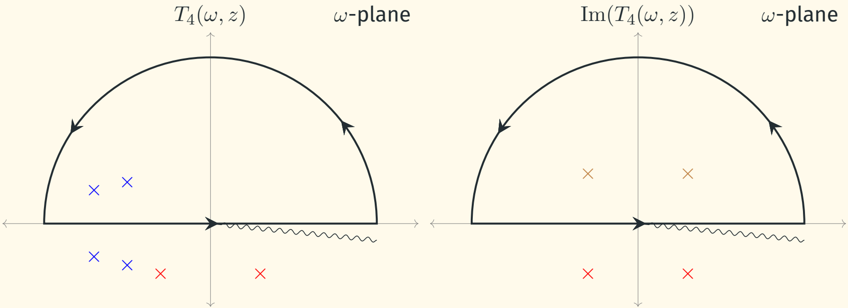
- Where $2\alpha := \Delta_{Tot.} - 3$ and

$$\mathcal{N} := \frac{\pi\lambda}{N \sin\theta}, \qquad \mathcal{N}_\alpha := \frac{(-2\mu^2)^\alpha}{2^{2+2\alpha}} \frac{\sin[(2\alpha+1)\theta]}{\sin(2\pi\alpha)}. \tag{6}$$

- Modified **Generalized Free Field** z-structure
- Satisfy "**permutation relations**" which relate different channels (and CFT crossing symmetry) [Lam Shao], [Mazáč] e.g.

$$z^{-2\Delta} f_t^{abcd}\left(\frac{1}{z}\right) = (1-z)^{-2\Delta} f_t^{adcb}(z) = (1-z)^{-2\Delta} f_t^{cbad}(z) \tag{7}$$

Unitarity and Optical Theorem



- **Resonances** mean we must consider more than just discontinuities when writing **Optical Theorem** in ω-plane
  - Separate generic amplitude into two parts $t_0 + t_{\log}$.
  - One sector has been discussed recently [Chang Huang Huang Li]

$$\mathrm{Im}(\mathcal{A}(\Delta_T, z)) = \text{[Chang Huang Huang Li Formula]} \tag{8}$$

$$+\#\sum_{\omega_{\mathrm{res}}}\left[\operatorname*{\mathbf{Res}}_{\omega=\omega^*_{\mathrm{res}}}\left(\omega^{\Delta_T-4}t_0^*(\omega, z)\right) + \operatorname*{\mathbf{Res}}_{\omega=-\omega_{\mathrm{res}}}\left(\omega^{\Delta_T-4}t_0(-\omega, z)\right)\right]$$

Conformal Block Decomposition

- Perform **conformal block decomposition** for CCFT 4pt func.
  - Two infinite towers of poles, "single trace" and "double trace"

$$\Delta_{\mathcal{O}} = n\,, \quad \Delta_{\mathcal{O}} = \alpha + n + 1\,, \quad n \in \mathbb{N}^0\,. \tag{9}$$

- **Conformal partial wave decomposition** by Euclidean IF
  - In 1d, conformal partial waves $\Psi_{\Delta,J}(z)$ have principal continuous series $\Delta = \frac{1}{2} + ir$, $r \in \mathbb{R}$ and discrete series.
  - Spurious poles of continuous series cancel against contributions from discrete series partial waves [Maldacena Stanford], [Simmons-Duffin, Stanford, Witten]
- Try to **factorize** CCFT 4pt function conformal block coefficients into $c^2_{\phi\phi\,\mathcal{O}}/d_{\mathcal{O}\,\mathcal{O}}$
  - Works for single trace tower. Must sum over both but have resonances $M^2_\pm$ with their distinct 3pt couplings $g^2_\pm$
  - Double trace operators do not factorize

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