# Duality Defects in $E_8$

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## The Chiral $E_8$ Theory

The **chiral** $E_8$ **theory** $V_{E_8}$ is the unique holomorphic CFT with central charge $c_L = 8$ and $c_R = 0$.

- Aka $E_8$ lattice VOA or chiral $(E_8)_1$ WZW model
- Boundary edge theory of Kitaev's $E_8$ phase

### Constructing the $E_8$ Theory

1. Start with the 8 dimensional $E_8$ lattice $L_{E_8}$.
2. For **every vector** $\alpha \in L_{E_8}$, we have a state $|\alpha\rangle$ created by the **vertex operator** $\Gamma_\alpha$. By definition $\langle\alpha|\beta\rangle = \delta_{\alpha\beta}$. Also
$$\Gamma_\alpha \Gamma_\beta = \epsilon(\alpha,\beta)\Gamma_{\alpha+\beta} \tag{1}$$
3. For **each root** we get a free **independent boson**, whose modes satisfy the commutation relations
$$[a^i_m, a^j_n] = m\delta^{ij}\delta_{m+n,0}\,. \tag{2}$$
4. Together they satisfy
$$a^i_n|\alpha\rangle = 0\,, \qquad \text{if } n > 0\,, \tag{3}$$
$$a^i_0|\alpha\rangle = \alpha^i|\alpha\rangle\,. \tag{4}$$

## Duality Defects

**Duality defects** are non-invertible topological defect lines which separate a theory from a dual theory.

- In our case, they are self-duality defects (analogous to **Kramers-Wannier** duality defect in Ising model) describing the isomorphism to a gauged theory
$$V_{E_8} \cong [V_{E_8}/\mathbb{Z}_m] \tag{5}$$
- Minimally enriches group-like fusion defect lines with
$$X_g \otimes \mathcal{N} = \mathcal{N} = \mathcal{N} \otimes X_g \tag{6}$$
$$\mathcal{N} \otimes \mathcal{N} = \bigoplus_g X_g \tag{7}$$
- This is a **Tambara-Yamagami category [TY]**

### Why Duality Defects?

- Archetypal non-invertible symmetry defect; requiring full formalism of fusion category symmetry.
- Provide **constraints on RG flows** (e.g. Ising Kramers-Wannier defect) **[CLSWY, GK, TW]**. Fusion category "rigidity" generalizes 't Hooft anomaly matching.
- Constrain modular bootstrap e.g. spin-selection rules constrain CFT spectrum **[CLSWY]**.
- Constraints for quantum gravity; **no global categorical symmetries** in quantum gravity.
- Modeling inhomogeneities in statistical physics.

## Additional Highlights from our Paper

- Review of LVOAs and contemporary results for enumerating their symmetries.
- Check our claims with: **fermionization** and mod 8 anomaly for $\mathbb{Z}_2$; embedding Potts model for $\mathbb{Z}_3$.
- Computer algorithm for enumerating duality defects and their partition functions
- More **associator data** for the TY categories
- A (2+1)d perspective on orbifolds and duality defects
- Proof for TY-line classification

## Classification and Construction

From fusion rules, we see if we encircle a local operator
$$\mathcal{N}\,\bigcirc\!\!\!\!\!\!\;\mathcal{O} = \begin{cases} \pm\sqrt{m}\,\mathcal{O} & \text{if } \mathcal{O} \text{ is uncharged} \\ 0 & \text{if } \mathcal{O} \text{ is charged} \end{cases}$$

- So, for example, a **twisted partition function** is
$$Z_{V_{E_8}}[0,\mathcal{N}] \sim \sum_{\text{uncharged}} \overset{?}{\pm}_{\mathcal{O}} |\chi_{\mathcal{O}}|^2 \tag{8}$$
- **Idea:** Track this sign as an additional $\mathbb{Z}_2$-twist acting on the invariant/shared subsector of $V_{E_8} \cap [V_{E_8}/G] = V_{E_8}^G$

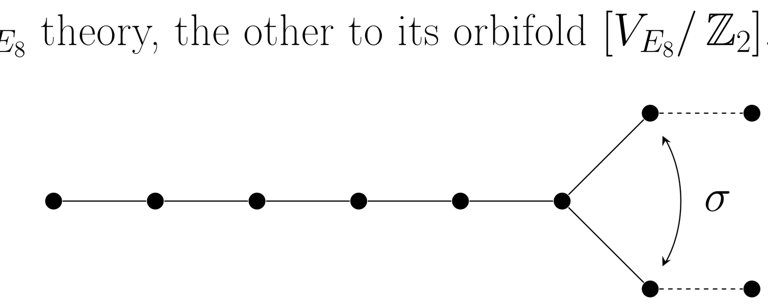
### Duality Defect Construction

To compute the $\mathbb{Z}_m$-duality defected partition function with $\mathcal{N}$ twisting Euclidean time $Z_{V_{E_8}}[0,\mathcal{N}]$:

1. Find all (conjugacy classes of) $\mathbb{Z}_m$ automorphisms of the $E_8$ Lie algebra by **Kac's Theorem**.
   - Compute which symmetries of $V$ are non-anomalous and thus can actually be gauged.
   - Note: there may be multiple $\mathbb{Z}_m$ conjugacy classes of non-anomalous symmetries
2. Pick a non-anomalous symmetry group $G \cong \mathbb{Z}_m$, the fixed-point sub-VOA $V_{E_8}^G$ is the lattice VOA of uncharged vertex operators from $(L_{E_8})^G$
   - The irreps $A := \mathrm{Irr}(V_{E_8}^G)$ form a metric Abelian group $(A,h)$, with $A \cong \mathbb{Z}_m \times \hat{\mathbb{Z}}_m$ and metric function $h(i,j) = ij/m$.
   - $V_{E_8}$ and $[V_{E_8}/G]$ decompose over the isotropic subgroups **along the axes** with irrep labels $(i,0)$ and $(0,j)$ respectively, i.e.
$$V_{E_8} = \bigoplus_{i\in\text{Electric}} \mathcal{H}^i_0\,, \tag{9}$$
$$[V_{E_8}/G] = \bigoplus_{j\in\text{Magnetic}} \mathcal{H}^0_j\,. \tag{10}$$
3. Finding automorphisms of $V_{E_8}^G$ which "swap the axes" corresponding to $V_{E_8}$ and $[V_{E_8}/G]$ in $A$, *we obtain the defected partition functions as the twisted characters for this second automorphism.*
$$Z_{V_{E_8}}[0,\mathcal{N}_{\hat{\sigma}}] = \mathrm{tr}_{V_{E_8}^G}\,\hat{\sigma} q^{L_0 - \frac{c}{24}}\,. \tag{11}$$

**Ex. $\mathbb{Z}_2$ Tambara-Yamagami Symmetries**

- There are two $\mathbb{Z}_2$ symmetries of $E_8$ up to conjugacy. One is non-anomalous and leaves a $D_8$ invariant, the other is anomalous leaving $A_1 \times E_7$ invariant.
- From the Dynkin diagram, we see that there are two ways to extend the $D_8$ lattice to an $E_8$ lattice. One corresponds to the original $V_{E_8}$ theory, the other to its orbifold $[V_{E_8}/\mathbb{Z}_2]$.

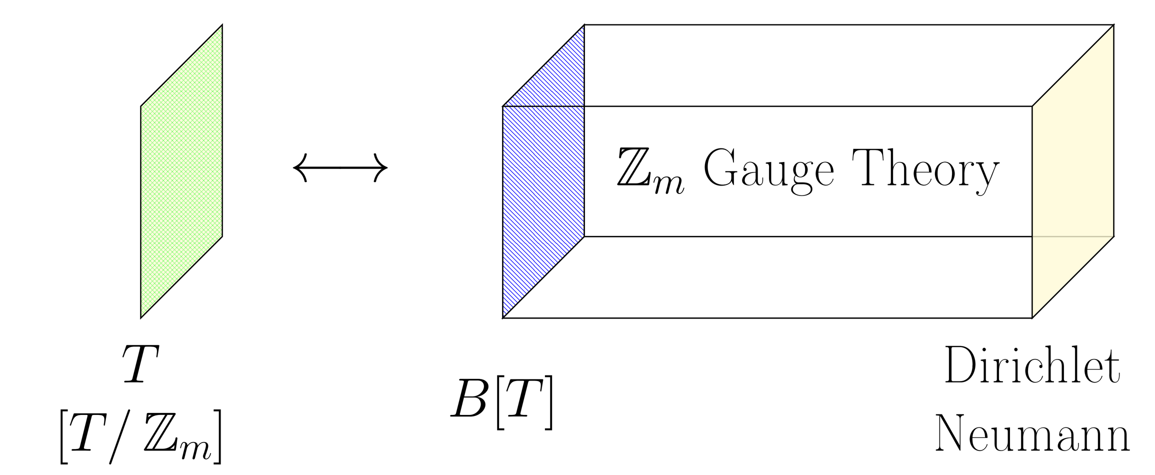



- These $E_8$'s are swapped by the lattice automorphism $\sigma$.
- The lattice automorphism $\sigma$ has 4-lifts to $\mathbb{Z}_2$ automorphisms of the VOA, so *there are 4 $\mathbb{Z}_2$ Tambara-Yamagami lines!*

## A (2+1)d TFT Interpretation

**Gauging 2d theories** has a (2+1)d interpretation in terms of topological gauge theories **[GK]**

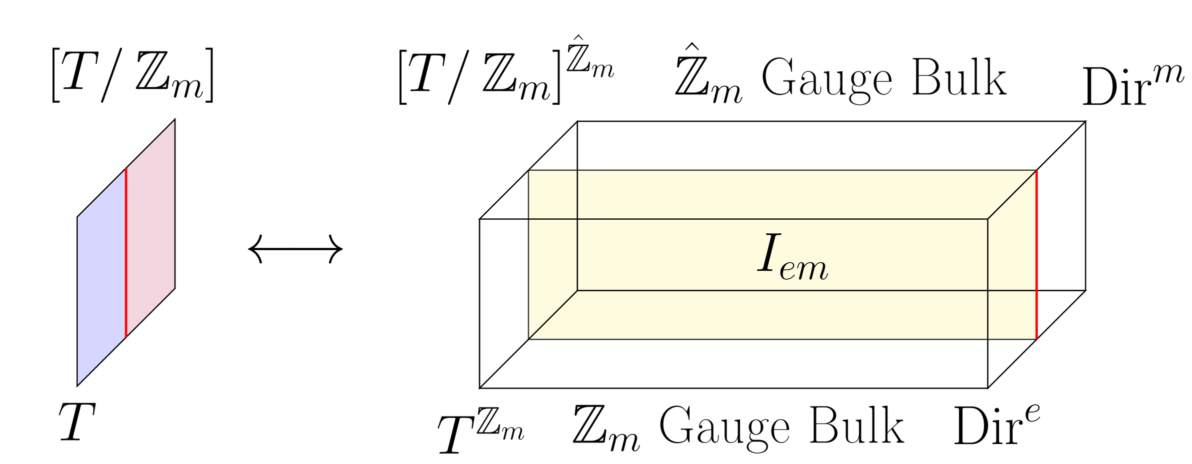
1. A 2d theory $T$ with a $\mathbb{Z}_m$ symmetry gives an "enriched Neumann" boundary condition $B[T]$ for a (2+1)d $\mathbb{Z}_m$ Dijkgraaf-Witten theory
2. $T$ and $[T/\mathbb{Z}_m]$ can be obtained by compactifying a segment with different boundary conditions.
   - The original theory $T = {}_{B[T]}[0,1]_{\text{Dir.}}$
   - The gauged theory $[T/\mathbb{Z}_m] = {}_{B[T]}[0,1]_{\text{Neu.}}$



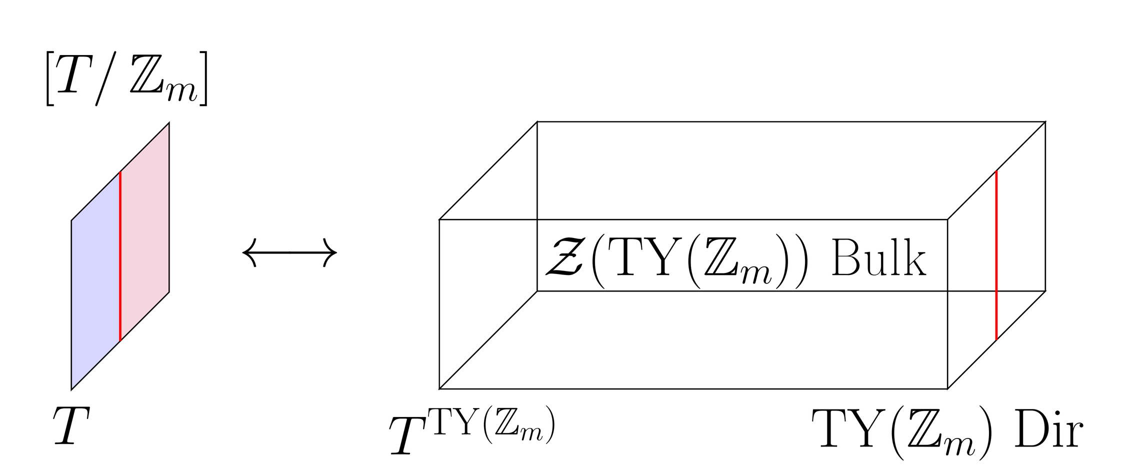

3. The Dirichlet and Neumann boundary conditions of the $\mathbb{Z}_m$ gauge theory are related by a $\mathbb{Z}_2^{em}$ 0-form symmetry *of the bulk TFT.*
   - Also called **electric-magnetic duality**
   - Corresponds to a codimension-1 interface $I_{em}$ in the TFT.

In (2+1)d, a **duality defect** becomes the endpoint (a **twist-line**) of an electric-magnetic duality interface $I_{em}$ in the bulk

- The theory with $\mathcal{N}$ inserted can be blown up to a slab of electric and magnetic $\mathbb{Z}_m$ gauge theory separated by the duality interface




- Can gauge the $\mathbb{Z}_2^{em}$ 0-form symmetry generated by the duality wall $I_{em}$, with Neumann boundary conditions for $\mathbb{Z}_2^{em}$ on the left and Dirichlet on the right.
  - This makes the duality wall $I_{em}$ "become invisible," leaving a $\mathcal{Z}(\mathrm{TY}(\mathbb{Z}_m))$ bulk.
  - We can gauge the emergent 1-form symmetry, i.e. **anyon condensation**, to go back to $\mathbb{Z}_m$ gauge theory.




## Tambara-Yamagami Actions on Holomorphic VOAs

Let $V$ be a holomorphic VOA. The inclusion $(V^{\mathbb{Z}_m})^{\mathbb{Z}_2} \subset V$ corresponds to a $\mathbb{Z}_m$ Tambara-Yamagami action on $V$ when the $\mathbb{Z}_2$ action swaps the axes of the metric Abelian group $\mathrm{Irr}(V^{\mathbb{Z}_m})$.

## Paper and Further References

Poster based on "Duality Defects in $E_8$" by I. M. Burbano, Justin Kulp, and Jonas Neuser

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**[TW]** R. Thorngren and Y. Wang, *Fusion Category Symmetry I: Anomaly In-Flow and Gapped Phases*, arXiv:1912.02817.

**[TY]** D. Tambara and S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, Journal of Algebra 209 (1998), no. 2 692–707