Holomorphic QFTs : Higher Structures and Bootstrap

Kasia Budzik, Davide Gaiotto, Justin Kulp,
Brian Williams, Jingxiang Wu, Matthew Yu.

PERIMETER INSTITUTE

Strings Vienna 2022 18-22 July

Holomorphic QFTs

Holomorphic QFTs depend only on the complex structure of the underlying spacetime manifold.

- Local operators endowed with structure of a **holomorphic factorization algebra [CG12]**.
- No UV divergences at one-loop.

Ex. **Toy Example: 2d Chiral Algebra**

- Free theories are clearly holomorphic due to the factorization of the 2d wave equation $\partial\bar{\partial}X(z,\bar{z}) = 0$.
- Closure of chiral OPE means (relative) holomorphic position dependence **persists at the quantum level**.
- Holomorphic factorization algebras provide higher analogues of 2d vertex/chiral algebra structures!

Holomorphic Twists

Given any **SUSY QFT**, we perform the **holomorphic twist** by passing to cohomology of one nilpotent supercharge; $Q := Q_-$ for example **[Joh]**.

- Anti-holomorphic translations are Q-exact; so the twisted theory is holomorphic in cohomology:

$$\{Q, \bar{Q}_{\dot{\alpha}}\} = \partial_{\bar{z}^{\dot{\alpha}}}\,. \tag{1}$$

- In BV-BRST formalism, simply deform BRST differential d by Q and take the cohomology with respect to

$$d_Q = d + Q\,. \tag{2}$$

- Different choices of twisting Q are captured by the **nilpotence variety**. For 4d $\mathcal{N}=1$, these are choices of complex structure on $\mathbb{C}^2$ **[IW1]**.

But Why (Holomorphically) Twist?

- Generally, twists restrict space of physical observables to more tractable BPS subset.
  - Compute protected quantities and **probe dualities**.
- The holomorphic twist is:
  - **Widely Available.** Need only an even dimensional theory and a nontrivial Maurer-Cartan element.
  - **Least Forgetful.** All other twists can be obtained by further deformation of holomorphic twist .

$$\text{Top. Twist} \subset \text{Holo. Twist} \subset \text{SUSY QFT} \tag{3}$$

- Operators captured by holomorphic twist are those counted by **superconformal index [IW2]**.

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{j_1+j_2-r/2} q^{j_1-j_2-r/2} e^{-\beta\{Q_-, S^-\}} \tag{4}$$

- **Infinite dimensional L_∞ symmetry** enhancements for global symmetries, analogous to 2d Virasoro and Kac-Moody
  - "Higher Virasoro" has two central charges, corresponding to a **and** c **anomalies** of original 4d CFT.

Holomorphic Descent

In twisted theory, identify superspace coordinates $\bar{\theta}^{\dot{\alpha}} \sim d\bar{z}^{\dot{\alpha}}$, so superfields are $(0,*)$-Dolbeault forms.

- Components of $(0,*)$-form $\mathcal{O}$ are obtained from $(0,0)$-form $\mathcal{O}^{(0)}$ by **holomorphic descent**

$$\mathcal{O}[\bar{\theta}] = e^{\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}}\mathcal{O}^{(0)} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)}\,. \tag{5}$$

- Call a superfield **semi-chiral** if

$$(Q + \bar{\partial})\,\mathcal{O} = 0\,. \tag{6}$$

  - Descendants $\mathcal{O}^{(k)}$ satisfy **descent equations**

$$Q\,\mathcal{O}^{(k)} + \bar{\partial}\,\mathcal{O}^{(k-1)} = 0\,. \tag{7}$$

Infinite Symmetries

Obtain **large family of symmetries** by integrating a semi-chiral $\mathcal{O}$ against any $\rho \in H^{2,*}_{\bar{\partial}}(\mathbb{C}^2 - 0)$ to obtain analogue of modes in 2d vertex algebra:

$$\hat{\mathcal{O}}_\rho = \oint_{S^3} \rho \wedge \mathcal{O}\,. \tag{8}$$

- $H^{2,*}_{\bar{\partial}}(\mathbb{C}^2 - 0)$ is concentrated in degrees 0 and 1.
  - °0 Has classes $\rho \sim z_1^n z_2^m d^2 z$ where $n, m \geq 0$. These give analogues of non-negative modes $\hat{\mathcal{O}}_{n,m}$.
  - °1 Has classes $\rho \sim \partial^n_{z_1}\partial^m_{z_2}\omega_{\mathrm{BM}}$ where $n, m \geq 0$ and ω_{BM} is the Bochner-Martinelli kernel (the free propagator). These give analogues of negative modes $\hat{\mathcal{O}}_{-n-1,-m-1}$.

The λ-Bracket

The non-negative modes above may be combined into a **λ-Bracket** (or **secondary product**):

$$\{\mathcal{O}_1, \mathcal{O}_2\}_\lambda = \oint_{S^3} e^{\lambda\cdot z} d^2 z\; \mathcal{O}_1(z,\bar{z})\,\mathcal{O}_2(0)\,. \tag{9}$$

- Product is **local**: can shrink S^3 without changing homology class of integration cycle **[OY]**.
- λ can be viewed as a holomorphic momentum.
- We recover the collision product when $\lambda = 0$.

Higher Brackets

The brackets we obtain endow the space of local operators with various algebraic structures: dga, dg Lie, Poisson etc. **in cohomology**.

- Existence of products is controlled by cycles in the **configuration space** of 2 points $\mathrm{Conf}_{\mathbb{C}^2}(2)$.
- Failure of these structures to survive at the **level of chains** can be measured by **higher brackets**

$$\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{n+1}\}_{\lambda_1,\ldots,\lambda_n} \tag{10}$$

  - Higher brackets are controlled by $\mathrm{Conf}_{\mathbb{C}^2}(n)$ with $n > 2$.
  - $(n+1)$-bracket describes homotopy between n-brackets

Ex. **Lie Algebras and L_∞-Algebras**

- An L_∞ algebra is a dg Lie algebra where the Jacobi identity only has to hold up to homotopy.

From Free to Interacting

Deforming a free theory to an interacting one by the interaction $\mathcal{I}$ changes the BRST differential; spoiling Q-closedness of certain operators.

- **Homotopy Transfer**. View $Q = Q_{\text{free}} + Q_{\text{int}}$ and compute Q-cohomology as cohomology of a differential acting on Q_{free}-cohomology H_{free}.

Perturbative Corrections

All perturbative corrections are contained in the free holomorphic factorization algebra!

- Take an operator $\mathcal{O}$ in the Q_{free}-cohomology and act with perturbative differential, we find perturbative corrections take the form of higher brackets of the free theory with $\mathcal{I}$:

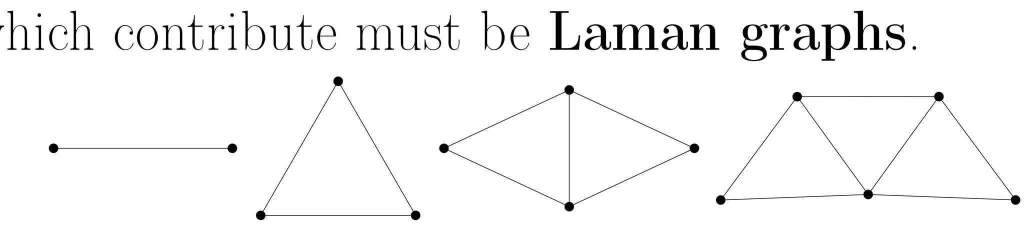
$$\begin{aligned} Q\,\mathcal{O} = {} & \{\mathcal{I},\mathcal{O}\}_0 + \{\mathcal{I},\mathcal{I},\mathcal{O}\}_{0,0} \\ & + \{\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{O}\}_{0,0,0} + \ldots\,. \end{aligned} \tag{11}$$

  - [Tree-level] $\sim 1\,\mathcal{I}$. [1-Loop] $\sim 2\,\mathcal{I}$'s. etc.
  - Deformations from higher Maurer-Cartan equation

Strategy: compute Feynman diagrams; obtain factorization algebra; deform by MC elements!

Bootstrapping from Laman Graphs

By counting form degrees, the Feynman graphs which contribute must be **Laman graphs**.



- **Configuration spaces** of graphs satisfy infinite collection of geometric **quadratic identities**; enforcing associativity graph-by-graph.

$$\sum_S \sigma(\Gamma, S)\,\mathcal{I}_{\Gamma[S]}[\lambda + \partial_{z'}; z]\,\mathcal{I}_{\Gamma(S)}[\lambda'; z'] = 0\,.$$

- We find that symmetries and quadratic identities are sufficients to **bootstrap Feynman integrals** to at least 3-loops (perhaps further!)

A UV/IR-Finite Master Integral and Bootstrap

All perturbative corrections to brackets are obtained by evaluating the **master integral**:

$$\mathcal{I}_\Gamma[\lambda; z] \equiv \int_{\mathbb{R}^{4|\Gamma_0|-4}} \bar{\partial}\left[\prod_{e\in\Gamma_1}\mathcal{P}(x_{e_0} - x_{e_1} + z_e, \bar{x}_{e_0} - \bar{x}_{e_1})\right]\left[\prod_{v\in\Gamma_0'} e^{\lambda_v\cdot x_v} d^2 x_v\right]. \tag{12}$$

Allows for arbitrary holomorphic shifts z_e, allowing study of **multi-local operators**. Aforementioned **quadratic identity** allows recursive bootstrap calculation from 1-loop result!

Paper and Further References

Based on multiple coming works, to appear here:

[Cos] K. Costello, *Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4*, arXiv:1111.4234

[CG12] K. Costello, O. Gwilliam, *Factorization algebras in quantum field theory*, Vol 1 & 2, New Mathematical Monographs. Cambridge University Press, Cambridge

[IW1] I. Saberi, B. Williams, *Twisted characters and holomorphic symmetries*, arXiv:1906.04221

[IW2] I. Saberi, B. Williams, *Superconformal algebras and holomorphic field theories*, arXiv:1910.04120

[Joh] A. Johansen, *Twisting of $\mathcal{N}=1$ supersymmetric gauge theories and heterotic topological theories*, arXiv:hep-th/9403017

[OY] J. Oh, J. Yagi, *Poisson vertex algebras in supersymmetric field theories*, arXiv:1908.05791

For applications to pure 4d $\mathcal{N}=1$ $SU(N)$ gauge theory, *holomorphic confinement*, and a holographic realization in the topological B-model, see the poster by Kasia Budzik.