Holomorphic Twists of SUSY QFTs

Graduate Students' Conference

Justin Kulp
with Kasia Budzik, Davide Gaiotto, Brian Williams, Jingxiang Wu and Matthew Yu.

Perimeter Institute for Theoretical Physics

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Introduction

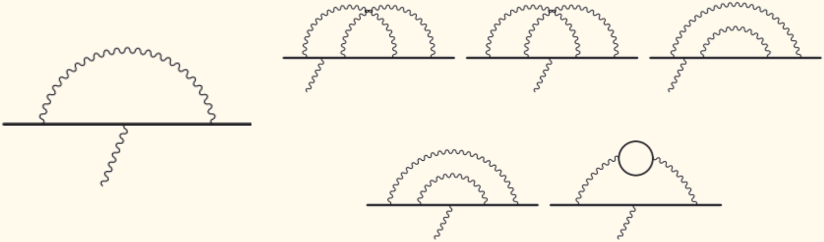
QFT is the Best!

- **Quantum Field Theory** (QFT) is the **most successful** scientific framework to date
  - Correct predictions about nature to an accuracy of 1 in 10^{10}

$$\mu = g\frac{e}{2m}\,,$$

$$a_{\mathrm{e}}^{\mathrm{theory}} = 0.001\,159\,652\,181\,\ldots\,, \qquad (1)$$

$$a_{\mathrm{e}}^{\mathrm{experiment}} = 0.001\,159\,652\,181\,\ldots\,. \qquad (2)$$



- QFT is **ubiquitous**: particle physics, condensed matter, cosmology, optics, quantum computing, quantum gravity.

QFT IS THE WORST!

What is QFT?

Despite QFTs triumphs, we ***do not know what QFT is***.

- QFT inspires incredible modern mathematics, yet has **no mathematically rigorous formulation**.
- Calculations traditionally done using perturbation theory

$$\mathcal{A} = \mathcal{A}_0 + g\,\mathcal{A}_1 + g^2\,\mathcal{A}_2 + \dots \qquad |g| \ll 1 \tag{3}$$

  - In electron magnetic moment calculation $g = \alpha \sim \frac{1}{137}$.
  - When interactions are **strong** (e.g. low-energy QCD), perturbation theory fails or misses crucial non-perturbative effects. *How do we study strongly coupled systems?*
- When perturbation theory works, enormous calculations can **unexpectedly simplify** to tiny answers. Hidden structure?
- Our most successful and ubiquitous tool is a black box!

Goals and Outline

(Life) Goals

Answer: What is Quantum Field Theory (QFT)?

- Develop new mathematical tools for QFT
- Obtain results at strong coupling
- Understand the space and structure of QFTs

Today we will focus on

1. Dualities and Supersymmetry
2. Holomorphic Twists
3. Interacting theories, Feynman diagrams, and confinement

and see how they address the problems above.

Dualities and Supersymmetry

Dualities 1/2

- One of our best tools for probing these problems is **duality**
  - Dualities are (often very non-trivial) equivalences between two different framings of the same underlying physics.

Ex. **Electromagnetic Duality** in vacuum $(\vec{E}, \vec{B}) \mapsto (\vec{B}, -\vec{E})$

$$\nabla \cdot \vec{E} = 0\,, \qquad \nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \qquad \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$$

Ex. **Jordan-Wigner Transformation** Non-local map from spin chain to fermionic chain

$$\sigma_i^z = 1 - 2c_i^\dagger c_i\,, \quad \sigma_i^+ = \prod_{j<i}(1-2c_j^\dagger c_j)c_i\,, \quad \sigma_i^- = \prod_{j<i}(1-2c_j^\dagger c_j)c_i\,,$$

Ex. **AdS/CFT** relates 4d $\mathcal{N} = 4$ Super Yang-Mills to type IIB string theory on $\mathrm{AdS}_5 \times S_5$.

Dualities 2/2

Ex. **Kramers-Wannier Duality** in Ising Model is a non-local map between high temperature and low temperature expansions in the same theory $K = -\frac{1}{2}\ln\tanh K'$

$$S_{\text{Ising}} = -K \sum_{\text{links}} \sigma_i \sigma_j \qquad \leftrightarrow \qquad S'_{\text{Ising}} = -K' \sum_{\text{links}} \sigma'_i \sigma'_j \tag{4}$$

- Dualities like the Ising example are particularly valuable because they relate **strong** and **weak couplings**

$$g' \sim \frac{1}{g} \tag{5}$$

  - Strong-weak dualities trade descriptions where quantum fluctuations are **large** for one where fluctuations are **small**

- Dualities $\rightsquigarrow$ understand space of QFTs and strong coupling.

SUSY Algebra

- Physics has **Poincaré symmetry** $\mathbb{R}^{1,d-1} \rtimes SO(1,d-1)$
  - Infinitesimally, there are spacetime rotations $M_{\mu\nu}$ and translations P_ρ satisfying

$$[M_{\mu\nu}, M_{\lambda\rho}] = 2\eta_{\nu[\rho}M_{\lambda]\mu} - 2\eta_{\mu[\rho}M_{\lambda]\nu}\,, \tag{6}$$

$$[M_{\lambda\rho}, P_\mu] = \eta_{\mu\rho}P_\lambda - \eta_{\mu\lambda}P_\rho\,, \tag{7}$$

$$[P_\mu, P_\nu] = 0\,. \tag{8}$$

- **Supersymmetry** (SUSY) adds anti-commuting generators

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}}P_m\delta^A_B\,, \tag{9}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = 0\,, \tag{10}$$

$$[P_m, Q^A_\alpha] = [P_m, \bar{Q}_{\dot{\alpha}A}] = 0\,, \tag{11}$$

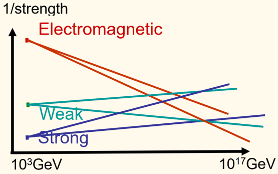
$$[P_m, P_n] = 0\,. \tag{12}$$

  - SUSY is a symmetry between **bosons** and **fermions**

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle\,, \qquad Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,. \tag{13}$$

SUSY QFT

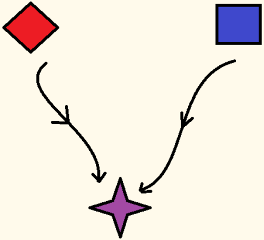
- But why study SUSY theories?
  - Phenomenological consequences: hierarchy problem, dark matter, Coleman-Mandula theorem, ...



- SUSY allows one to gain insights into pure QFT
  - **Monotonen-Olive duality**.

$$\begin{array}{c}\text{4d } \mathcal{N}=4 \text{ SYM} \\ (G,\alpha)\end{array} \leftrightarrow \begin{array}{c}\text{4d } \mathcal{N}=4 \text{ SYM} \\ ({}^{L}G, \frac{1}{\alpha})\end{array}$$

    Which fields are fundamental?
  - **Seiberg duality**. SQCD with gauge groups $SU(N_c)$ and $SU(N_f - N_c)$ can flow to the same IR fixed point. Gauge symmetry is not fundamental.



  - **Non-renormalization theorems**. E.g. in 4d $\mathcal{N}=2$ there are no perturbative corrections to the β-function beyond one-loop.

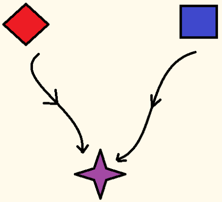
Holomorphic Twists

BPS Quantities

- What makes SUSY so powerful and tractable (in part) is the existence of **BPS operators**
  - These are operators $\mathcal{O}$ annihilated by one of the supercharges

$$\{Q^A_\alpha, \mathcal{O}\} = 0 \qquad (14)$$

  - BPS states often saturate a number of interesting conditions.
- The BPS states created by these BPS operators form **short multiplets**
  - A "standard" multiplet has $2^{2\mathcal{N}}$ states. These multiplets are "short" because of (14)
  - Because dimension of a representation can't change continuously, BPS states are **protected** from quantum corrections and continuous deformations



- **Twisting** restricts the physical theory to a BPS subset.
  - The twisted theory is completely **mathematically rigorous**.
  - Check (and prove) dualities at the level of BPS quantities.
  - Conjectural dualities lead to powerful mathematics.

To twist, pick any supercharge Q.

1. First note that $Q^2 = 0$.
2. In the twist, we will take Q-invariants $[Q, \mathcal{O}] = 0$.
3. If we assume SUSY isn't broken (so that $Q\,|0\rangle = 0$) then a product of Q-invariant operators satsifies

$$\langle (\mathcal{O} + [Q, \Lambda]) \cdots \rangle = \langle \mathcal{O} \cdots \rangle + \langle [Q, \Lambda \cdots] \rangle = \langle \mathcal{O} \cdots \rangle \ , \tag{15}$$

   So doesn't care about operators modulo $\mathcal{O} \sim \mathcal{O} + [Q, \Lambda]$.

- The twist is the **Q-cohomology** of local operators.

$$Q^2 = 0\,, \quad Q\text{-Closed: } [Q, \mathcal{O}] = 0\,, \quad Q\text{-Exact: } [Q, \Lambda]\,. \tag{16}$$

Holomorphic Twists

- Recall the SUSY algebra bracket

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \,, \tag{17}$$

  - Our choice of Q selects **two directions** to be **Q-exact**.
  - Correlation functions don't depend on these two directions!
- 4d spacetime is $\mathbb{R}^4$, but we can also view this as $\mathbb{C}^2$
  - We can view these two directions as forming an anti-holomorphic direction $\bar{z}^{\dot{\alpha}}$ in $\mathbb{C}^2$
  - So the twisted theory is (cohomologically) holomorphic

$$\{Q, \bar{Q}_{\dot{\alpha}}\} = \partial_{\bar{z}^{\dot{\alpha}}} \tag{18}$$

- Thus we call this the **Holomorphic Twist**
  - If we asked for the cohomology with respect to another supercharge, we would have a **topological twist**

Holomorphic QFTs

- Just like **Topological QFTs** only care about the topology of spacetime, **Holomorphic QFTs** depend only on the complex structure of spacetime.
  - Has structure of **Holomorphic Factorization Algebra**.
  - Example. 2d Chiral Algebra and/or Vertex Algebra
- Infinite dimensional symmetry enhancements analogous to Virasoro and Kac-Moody.
- Theories are equipped with local product called **λ-Bracket**

$$\{\mathcal{O}_1, \mathcal{O}_2\}_\lambda = \oint_{S^3} e^{\lambda\cdot z} d^2z\, \mathcal{O}_1(z,\bar{z})\, \mathcal{O}_2(0) \tag{19}$$

  - Generalizing OPE in 2d CFT
  - **Higher brackets** describe homotopy between lower brackets

$$\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{n+1}\}_{\lambda_1,\ldots,\lambda_n} \tag{20}$$

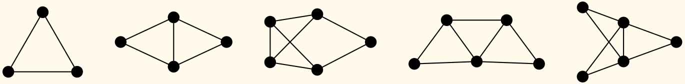
Interactions and Feynman Diagrams

- In an interacting quantum theory **all perturbative corrections** are contained in the higher brackets of the free holomorphic factorization algebra!

$$\mathbf{Q}\,\mathcal{O} = \{\mathcal{I},\mathcal{O}\}_0 + \{\mathcal{I},\mathcal{I},\mathcal{O}\}_{0,0} + \{\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{O}\}_{0,0,0} + \dots . \qquad (21)$$

  - [Tree-level] $\sim 1\,\mathcal{I}$, [1-Loop] $\sim 2\,\mathcal{I}$'s , etc.
  - The free theory knows about its deformations.

- Feynman diagrams in theory must be **Laman graphs**.



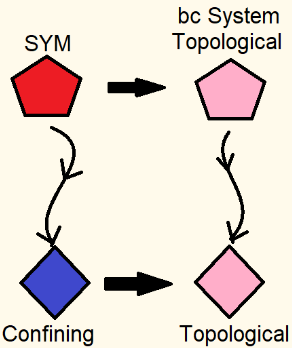
  - Arbitrary integral takes the form:

$$\mathcal{I}_\Gamma[\lambda;z] \equiv \int_{\mathbb{R}^{4|\Gamma_0|-4}} \bar{\partial}\left[\prod_{e\in\Gamma_1}\mathcal{P}(x_{e_0}-x_{e_1}+z_e,\bar{x}_{e_0}-\bar{x}_{e_1})\right]\left[\prod_{v\in\Gamma_0'} e^{\lambda_v\cdot x_v}d^2x_v\right] \qquad (22)$$

- Integrals controlled by polytope in space of loop momenta, the **operatope**. Can **bootstrap** all integrals from geometry.

Holomorphic Confinement

- **Confinement** of quarks is one of the enduring mysteries of modern physics.
  - Colour-charged objects must form bound states which are colour neutral.
  - $1M prize to prove Yang-Mills has a mass gap (and thus flows to a topological theory in the IR)
- Computing the **holomorphic twist** of $\mathcal{N} = 1$ super Yang-Mills we find it is miraculously **topological**!
  - Could this be a **novel UV manifestation of confinement**?!



Conclusion

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Recap:

1. Dualities and Supersymmetry
2. Holomorphic Twists
3. Interacting theories, Feynman diagrams, and confinement

(Life) Goals

Answer: What is Quantum Field Theory (QFT)?

- Develop new mathematical tools for QFT: **twist**, **higher Maurer-Cartan equation**, **Bootstrap**
- Obtain results at strong coupling: **holomorphic confinement**
- Understand the space and structure of QFTs: **dualities**, **holomorphic factorization algebra**, **operatope**

Fin