Holomorphic Twists and Confinement in N=1 SYM

for Harvard CMSA
Quantum Matter Seminar

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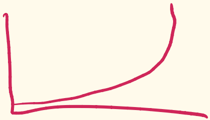
04/Oct/2022

arXiv:2207.14321, arXiv:221x.xxxxx, arXiv:221x.xxxxx

Outlines and Punchlines



I. **Introduction and Motivation.**
II. **Basics Definitions: SUSY and Twist.**
III. **Infinite Symmetries and Higher Brackets.**
IV. **Cohomology and Holomorphic Confinement of $\mathcal{N} = 1$ SYM.**
V. **Recap.**

Three Punchlines

1. Holomorphic twist of SQFTs
2. Infinite dimensional symmetries
3. Holomorphic Confinement of $\mathcal{N} = 1$ Super Yang-Mills (SYM)

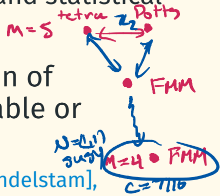
Introduction and Motivation

WHY SUPERSYMMETRY? 1/2

- **Supersymmetry** (**SUSY**) is a symmetry between bosons and fermions

$$Q\left|\text{boson}\right\rangle = \left|\text{fermion}\right\rangle\,, \qquad Q\left|\text{fermion}\right\rangle = \left|\text{boson}\right\rangle\,. \tag{1}$$

- Particle phenomenology and experiment
  - Coleman-Mandula theorem, hierarchy problem, dark matter
  - Emergent supersymmetry in condensed matter and statistical systems [Lee]++, [Kikuchi, Chen, Xu, Chang]++

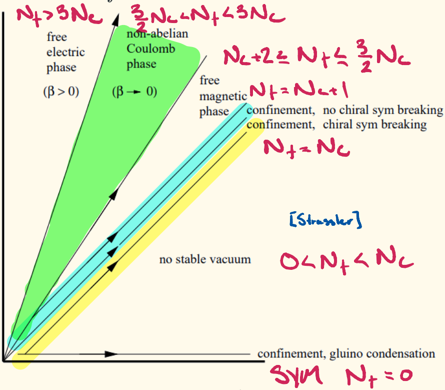


- SUSY QFTs provide a rich, but tractable, collection of theories in which quantities are exactly computable or **highly constrained**
  - Non-renormalization theorems [Sohnius, West], [Mandelstam], [Grisaru, Rocek, Siegel], [Seiberg], [Argyres, Plesser, Seiberg] ++
  - Electric-magnetic duality [Montonen, Olive], Seiberg-Witten Theory [Seiberg, Witten], Seiberg duality [Seiberg]

- Insights from SUSY QFTs can tell us about the real world
  - Directly or by studying SUSY breaking [Dine, Yu] ++

Ex. **Super QCD.** $\mathcal{N} = 1$ $SU(N_c)$ SYM with N_f flavors. [Seiberg]++

- Tells us which properties of $N_f = N_c = 3$ are generic
- $N_f = 0$ **(SYM).** **Confinement** and **chiral symmetry breaking** (CSB).
- $N_f = N_c$ Confinement and CSB
- $N_f = N_c + 1$ Confinement and NO CSB
- Strongly coupled IR CFTs...



- Insights from SUSY tell us about pure mathematics
  - Supersymmetry and Morse theory [Witten]

Why Twist?

- The tractability of SUSY QFTs comes, in part, from the existence of various **protected quantities**
  - Can be invariant under deformation of coupling constant
  - Computable in different duality frames; probe NP physics

  Ex. F-terms and chiral operators. Protected by two of the four supercharges. Constrain the space of supersymmetric vacua and deformations of the theories [Seiberg].

  Ex. Supersymmetric index counts local operators with signs [Kinney, Maldacena, Minwalla, Suvrat], [Romelsberger], [Dolan, Osborn]

- **Twisting** isolates these protected quantities [Witten]
  - The twisted theory is completely **mathematically rigorous**.
  - Check (and prove) dualities at the level of BPS quantities.
  - Conjectural dualities lead to powerful mathematics.

The Punchlines in Context

1. Holomorphic twist of SQFTs
   - Holomorphic twist captures the protected quantities which give SUSY QFT its power
   - Calculations in twisted theory are tractable and constrained
   - Mathematically rigorous setting
2. Infinite dimensional symmetries
   - Subsector of the original undeformed theory has infinite symmetry enhancements
3. Holomorphic Confinement of $\mathcal{N} = 1$ Super Yang-Mills (SYM)
   - Twist provides novel approach to understanding phases of SQFTs

Basic Definitions: SUSY and the Holomorphic Twist

SUSY and the Semi-Chiral Ring

- **Supersymmetry** enhances Poincaré symmetry

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = \delta^A_B P_{\alpha\dot{\beta}}\,, \tag{2}$$

$$\{Q^A_\alpha, Q^B_\beta\} = \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = 0\,. \tag{3}$$

  - In Euclidean signature, $\mathrm{Spin}(4) \cong SU(2)_L \times SU(2)_R$
  - Q_α and $\bar{Q}_{\dot{\alpha}}$ are two-component spinors
- Pick some supercharge $Q = Q_-$
  - The **semi-chiral ring** of Q consists of all Q-invariant operators

$$[Q, \mathcal{O}] = 0 \tag{4}$$

  - If SUSY isn't broken (so that $Q\,|0\rangle = 0$), then a product of Q-invariant operators satisfies

$$\langle(\mathcal{O} + [Q, \Lambda])\cdots\rangle = \langle\mathcal{O}\cdots\rangle + \langle[Q, \Lambda\cdots]\rangle = \langle\mathcal{O}\cdots\rangle\;, \tag{5}$$

    So doesn't care about operators modulo $\mathcal{O} \sim \mathcal{O} + [Q, \Lambda]$.

- Given any SQFT, we obtain the **Holomorphic Twist** by taking **cohomology** of any one nilpotent supercharge, e.g. $Q := Q_-$

2d VA
Frobenius (TFT)

$$Q^2 = 0\,, \quad Q\text{-Closed: } [Q, \mathcal{O}] = 0\,, \quad Q\text{-Exact: } [Q, \Lambda]\,, \tag{6}$$

  - Most available & least forgetful twist: only needs $\mathcal{N} = 1$ SUSY.
  - Cohomology isolates the semi-chiral ring

- Anti-holomorphic translations are Q-exact, so twisted theory is (cohomologically) **holomorphic** [Johansen], [Nekrasov], [Costello]

$$\{Q, \bar{Q}_{\dot\alpha}\} = \partial_{\bar{z}^{\dot\alpha}} \tag{7}$$

$\mathbb{R}^4 \simeq \mathbb{C}^2$

- Operators captured by holomorphic twist are those counted by the **superconformal index** [Saberi, Williams]

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{j_1+j_2-r/2} q^{j_1-j_2-r/2} e^{-\beta\{Q_-, S_+\}} \tag{8}$$

Superfields and Dolbeault Cohomology

- 4d $\mathcal{N} = 1$ SQFTs are formulated in language of superspace.
  - Spacetime is a supermanifold with odd coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$
  - SUSY multiplets become components of a superfield
- We use a **chiral superspace**, with only $\bar{\theta}^{\dot{\alpha}}$ coordinates
  - The right-handed supercharges act by $\bar{Q}_{\dot{\alpha}} = \partial_{\bar{\theta}^{\dot{\alpha}}}$
  - A superfield is of the form

$$O[\bar{\theta}] = e^{\bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}} O^{(0)} = O^{(0)} + O^{(1)} + O^{(2)} \tag{9}$$

- Identifying $\bar{\theta}^{\dot{\alpha}} \leftrightarrow d\bar{z}^{\dot{\alpha}}$ allows us to think of superfields as $(0, \bullet)$**-Dolbeault forms**
- A superfield is semi-chiral if it satisfies $(Q + \bar{\partial})\,\mathcal{O} = 0$, which happens iff its bottom component is a semi-chiral operator
  - The other components satisfy **holomorphic descent** relations

$$Q\,\mathcal{O}^{(n)} + \bar{\partial}\,\mathcal{O}^{(n-1)} = 0 \tag{10}$$

$\mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)}$

Infinite Dimensional Symmetries and (Higher) Brackets

A 2D CFT REMINDER

- If ϕ is a **2d chiral conformal primary** with chiral dimension h, then we can expand

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\hat{\phi}_n = z^{-h}\left(\cdots + z^{-1}\hat{\phi}_1 + z^0\hat{\phi}_0 + z^1\hat{\phi}_{-1} + \dots\right)$$

- If we want to **extract the mode** $\hat{\phi}_n$, we know that

$$\hat{\phi}_n = \oint \frac{dz}{2\pi i} z^{n+h-1}\phi(z) \tag{11}$$

  - Formally, extract $n \geq 0$ modes by multiplying by generator z^n for the space of holomorphic functions on $\mathbb{C}$
  - Extract $n < 0$ modes by integrating against Bochner-Martinelli kernel

$$\hat{\phi}_n = \frac{1}{n!}\phi^{(n)}(0) \propto \oint \partial^n \omega_{\mathrm{BM}}\phi(z) \tag{12}$$

    where $\omega_{\mathrm{BM}} = 1/z$

- These are elements of $H^{1,0}(\mathbb{C}^1\backslash\{0\})$. Note: $H^{1,1}(\mathbb{C}^1\backslash\{0\}) = 0$

Dolbeault Cohomology of $H^{n,\bullet}(\mathbb{C}^n \setminus \{0\})$

concentrated in deg 0 and n-1

Functions ↑ ↑ Dual

- In other words

$$\hat{\phi}_n = \int_{S^1} [z^h \phi(z) d\bar{z}] \wedge \rho \,, \quad \rho \sim \begin{cases} z^n dz \\ \partial^n \omega_{\mathrm{BM}} \end{cases} \in H^{1,0}(\mathbb{C} \setminus \{0\})\,. \tag{13}$$

- $H^{2,\bullet}(\mathbb{C}^2 \setminus \{0\})$ is **concentrated in degrees 0 and 1**
  - Degree 0. Classes are (2,0)-Dolbeault forms with $n, m \geq 0$:

$$\rho \sim z_1^n z_2^m d^2 z \tag{14}$$

  - Degree 1. Classes are (2,1)-Dolbeault forms with $n, m \geq 0$:

$$\rho \sim \partial_{z^1}^n \partial_{z^2}^m \omega_{\mathrm{BM}} \tag{15}$$

  - The **Bochner-Martinelli Kernel** is now the thing such that

$$\int_{S^3} \omega_{BM} f(z) = f(0) \tag{16}$$

for any holomorphic $f(z)$ on $\mathbb{C}^2$.

MODES IN THE HOLOMORPHIC TWIST

- In $\mathbb{C}^2$, we've seen that (chiral) superfields $\mathcal{O}$ are identifiable with $(0,\bullet)$-forms
- For any $\rho \in H^{2,\bullet}(\mathbb{C}^2 \setminus \{0\})$ we define

$$\hat{O}_\rho = \oint_{S^3} O \wedge \rho \tag{17}$$

  - Degree 0 classes $\rho \sim z_1^n z_2^m d^2 z$ give analogue of **non-negative modes** of VOA, $\hat{O}_{n,m} = \hat{O}_\rho$
  - Degree 1 classes $\rho \sim \partial_{z^1}^n \partial_{z^2}^m \omega_{\mathrm{BM}}$ give analogue of **negative modes** of VOA $\hat{O}_{-n-1,-m-1} = \hat{O}_\rho$
- **Infinite dimensional symmetry enhancements** analogous to Virasoro and Kac-Moody [Gwilliam, Williams]. [Saberi, Williams]
  - Deformation $\rightsquigarrow$ [Beem, Lemos, Liendo, Peelaers, Raselli, van Rees]
- Stress: Statement about the OPE of a subsector of the **original physical theory**, *not* a deformed/modified theory.

Summary of Symmetries

Supercurrent Multiplet $\quad S_{\alpha,\mu} \quad T_{\mu\nu}$

I define

$$S^{(0)}_{\dot\alpha} \equiv S_{+,+\dot\alpha}$$

It turns out

$$Q(\partial^{\dot\alpha} S^{(0)}_{\dot\alpha}) = 0$$

$\Leftrightarrow$ $\partial^{\dot\alpha} S_{\dot\alpha}$ is semi-chiral

Actually, when cons. R-sym

$S_{\dot\alpha}$ is semi-chiral

As it turns out, $S^{(1)}_{\dot\alpha}$ contains the holomorphic part of physical stress tensor and generates diffs of spacetime.

{Negative $\hat{S}_{n,m}$}

$\oplus$

Complex Symplectomorphisms i.e. Pres $d^2 z$ (Gen. by $\underset{+ve}{\hat{S}_{n,m}}$) $\subset$ Complex Diffeomorphisms (Needs R sym)

Higher Virasoro (brace around the above)

$\cup$ $\quad$ $\cup$

Global $su(2)$ subalgebra (Gen. by holomorphic part of $T^{\text{physical}}_{\mu\nu}$) $\subset$ Global $u(2)$ subalgebra (Needs R sym)

$SU(2) \longrightarrow U(2)$

Brackets and Higher Brackets

- Theories are equipped with local product called **λ-Bracket**

$$\{\mathcal{O}_1, \mathcal{O}_2\}_\lambda = \oint_{S^3} e^{\lambda \cdot z}\, \mathcal{O}_1(z, \bar{z})\, \mathcal{O}_2(0)\, d^2 z \tag{18}$$

  - Product is local since we can shrink S^3 without changing homology class of integration cycle
  - $\mathcal{O}_1$ and $\mathcal{O}_2$ have **regular OPE** if this bracket vanishes.
  - Tells you what happens to a SUSY transformation of O_2 if you add O_1 as an F-term (interaction)

- **Higher brackets** describe homotopy between lower brackets

$(\Theta_1\Theta_2)\Theta_3 = \Theta_1(\Theta_2\Theta_3)$ $\{\Theta_1, \Theta_2, \Theta_3\}$ = associator

$$\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{n+1}\}_{\lambda_1, \ldots, \lambda_n} \tag{19}$$

  - Capture operations failure to be holomorphic on chains.

Cohomology and Confinement

FREE COHOMOLOGY AND ADDING INTERACTIONS

- Start with the free **Free Cohomology** $\mathcal{V}$
  - Gauge-invariant polynomials (words) in fields and derivatives (letters)

- **Interacting quantum theory** is obtained from underlying free-classical theory $\mathcal{V}$ as cohomology of a new operator

$$\mathbf{Q} = Q_0 + Q_1 + Q_2 \ldots . \tag{20}$$

Chiral Analogue [Cachazo, Douglas, Seiberg, Witten]

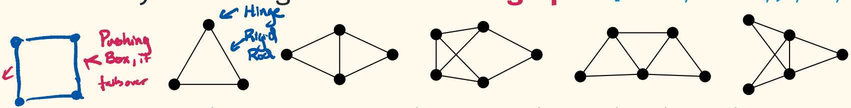
where Q_n is computed by n-loop Feynman diagrams.

- In interacting quantum theory **all perturbative corrections** are contained in the higher brackets of the free holomorphic factorization algebra! [Budzik, Gaiotto, JK, Williams, Wu, Yu]

$$\mathbf{Q}\,\mathcal{O} = \{\mathcal{I},\mathcal{O}\}_0 + \{\mathcal{I},\mathcal{I},\mathcal{O}\}_{0,0} + \{\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{O}\}_{0,0,0} + \ldots . \tag{21}$$

  - [Tree-level] $\sim$ 1 $\mathcal{I}$, [1-Loop] $\sim$ 2 $\mathcal{I}$'s , etc.

- Feynman diagrams are **Laman graphs**. [Budzik, Gaiotto, JK, Wu, Yu]



  - Arbitrary holomorphic-topological twists in arbitrary dimensions are captured by generalized-Laman graphs [Kontsevich], [Gaiotto, Moore, Witten]

- Arbitrary integral takes the form:

$$\mathcal{I}_\Gamma[\lambda; z] \equiv \int_{\mathbb{R}^{4|\Gamma_0|-4}} \bar{\partial} \left[\prod_{e \in \Gamma_1} \mathcal{P}(x_{e_0} - x_{e_1} + z_e, \bar{x}_{e_0} - \bar{x}_{e_1}) \right] \left[\prod_{v \in \Gamma_0'} e^{\lambda_v \cdot x_v} d^2 x_v \right] \tag{22}$$

  - Change of variables maps integral to Fourier transform of a polytope in loop momenta, the **operatope**.
  - Feynman integrals satisfy infinite collection of geometric **quadratic identities**; enforcing associativity
  - Can **bootstrap** all Feynman integrals from these identities.

Twisting $\mathcal{N} = 1$ SYM

- $\mathcal{N} = 1$ **SYM** is $SU(N)$ gauge theory with an adjoint fermion

$$\mathcal{L} = \int d^2\theta \frac{-i}{8\pi} \tau \,\mathrm{tr}\, W_\alpha W^\alpha + \mathrm{c.c.} \tag{23}$$

- Twist is identified (in a non-trivial way) with a **holomorphic** bc **system** [Costello], [Elliot, Safronov, Williams], [Saberi, Williams]
  - Fields are collected in adjoint bosonic superfield b and (co)adjoint fermionic superfield c.

- The Lagrangian of this theory is

$$\mathcal{L}_{\text{twisted}} = \mathrm{Tr}\, b \left(\bar{\partial} c - \frac{1}{2}[c, c] \right) + \tau\, \mathrm{Tr}\, \partial_\alpha c\, \partial^\alpha c \,. \tag{24}$$

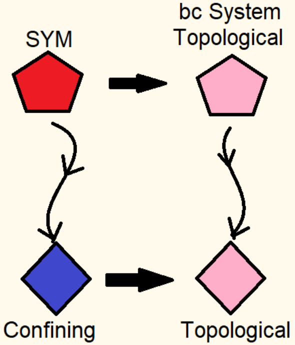
  - Free cohomology

$$\mathbb{C}[b, \partial_\alpha b, \partial_\alpha \partial_\beta b, \ldots, \partial_\alpha c, \partial_\alpha \partial_\beta c, \ldots]^G \tag{25}$$

  - Derivative of the stress tensor is $\partial_\alpha S^\alpha = \partial_\alpha b_A \partial^\alpha c^A$.

Holomorphic Confinement

- Adding **one loop corrections**, we find $\partial_\alpha S^\alpha = Q \, \mathrm{Tr}\, b^2$
  - $\partial_\alpha S^\alpha$ generates (among other things) all the remaining spacetime symmetries. SU(2)
  - Exactness of $\partial_\alpha S^\alpha$ means local operators are invariant under remaining spacetime transformations.
  - Theory becomes **topological** at one loop! ← Survives higher loop corrections



- Being topological is compatible with **confinement**: if topological in the UV, then topological in the IR.
  - Constrains IR physics: the holomorphic twist of the IR must also be topological.
  - We call this **Holomorphic Confinement** [Budzik, Gaiotto, JK, Williams, Wu, Yu]
- **Large N** calculations with **twisted holography** dual.

Recap

Conclusion

- SUSY QFTs are toy models for real world physics. Tractability comes from protected quantities.
  - Protected quantities are isolated by **the holomorphic twist**, by taking the cohomology of one supercharge.
  - The operators that survive are those that contribute to the **superconformal index**.
  - This is not a deformation, it is a statement about a subsector of the original theory.

Features

These sectors have:

1. **Infinite dimensional symmetry enhancements**, analogous to Virasoro and Kac-Moody symmetries
2. Feynman diagrams which are completely **bootstrappable**
3. **Novel UV manifestation of confinement**

Future Goals

- Super QCD. Twist has 4d $PSU(N_f|N_f)$ Kac-Moody symmetry.
- $\mathcal{N} = 4$ SYM. Compute cohomology of $\frac{1}{16}$-BPS states.
  - Can the superconformal index be written as a sum of characters for the infinite dimensional symmetry algebras?
- **Seiberg Duality**. Requires **non-perturbative corrections**.
  - Prove Seiberg duality at level of twisted theory.
  - What does Seiberg duality translate to for pure mathematics?
- Understand if the infinite dimensional symmetry algebras can be used for the bootstrap (like the $\mathcal{N} = 2$ VOAs)

Fin