Hyperbolic Honeycombs and Self-Similar Quasicrystals

for Quantum Fields & Strings Seminar

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Outlines and Punchlines

I. **Introduction and Motivation.**
II. **Beyond Euclidean Lattices.**
III. **Hyperbolic Quasicrystals 2d/1d.**
IV. **Higher Hyperbolic Quasicrystals.**
V. **Conclusion.**

Three Punchlines

1. Hyperbolic lattices and quasicrystals
2. New quasiperiodic patterns
3. Applications to physics (holography, condensed matter)

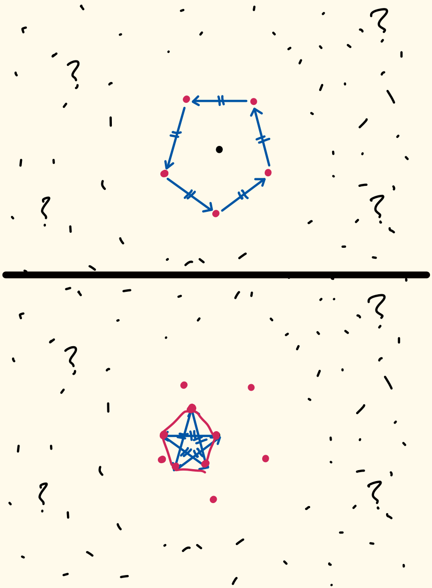
Introduction and Motivation

History: Patterns, Tessellations, and Lattices

- Sumerians, ancient greece
  - *What shapes must a unit have to fill space without gaps by figures congruent to that unit?*
- Point groups < $O(d)$
  - Infinitely many 2d point groups, $\mathbb{Z}_n$ and D_{2n}.
- Translational symmetry
  - **Crystallographic restriction theorem**
  - Only 2,3,4,6-fold symmetry around a point in 2d and 3d
  - 17 Wallpaper Groups in 2d.

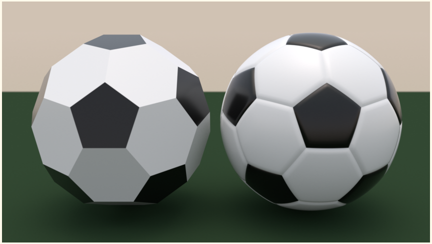
No 5-Fold Symmetry in 2D/3D

1. Pick a site of 5-fold symmetry
2. Take your favourite nearby lattice site and hit it with your 5-fold symmetry action.
3. Generates a regular pentagon of sites in a plane of the lattice.
4. All pentagon displacement vectors must be in the lattice.
5. Add them up at a single point to generate sites forming a smaller pentagon.

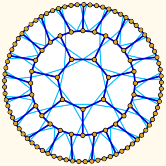


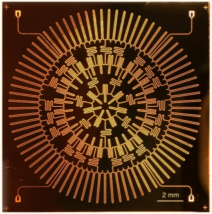
Hyperbolic Space

- We can tile other surfaces
  - Sphere (EdS) and hyperbolic space (EAdS) are other maximally symmetric spaces.
  - Tilings of sphere closely related to existence of 3d polyhedra



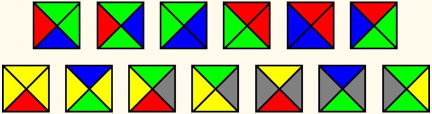
- Hyperbolic space is more novel
  - Harder to intuit
  - Applications in art, biology, network theory, and physics
  - Hyperbolic band theory, tensor networks, error-correction (HaPPY code), "discrete holography"





A Simple Question. A Crazy Answer.

- Given a set of **prototiles** τ with rule that colours must match, can you use them to tile $\mathbb{R}^2$?
  - Is there an algorithm to determine if τ tiles the plane?



- Options:
  1. τ does not tile $\mathbb{R}^2$.
  2. τ tiles $\mathbb{R}^2$ and admits a periodic tiling.
  3. τ tiles $\mathbb{R}^2$ but never admits periodic tilings. τ is an **aperiodic** set.

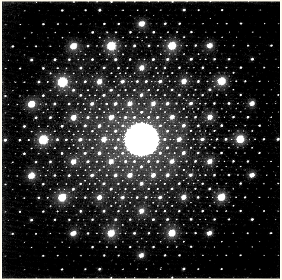
! There exists an algorithm to determine if τ tiles the plane iff all possible τ are of type 1 or type 2.

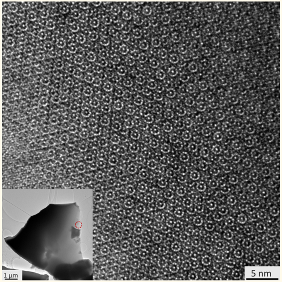
! There is a map from

$$\{\texttt{Turing Machines } M\} \mapsto \{\texttt{Tile Sets } \tau(M)\} \qquad (1)$$

such that M halts iff $\tau(M)$ tiles the plane.

- **Quasicrystals** are aperiodic tilings which still possess order.
  - **Aperiodic** means no translation symmetry by any amount at all.
- Possess classically **forbidden symmetries**.
  - Spoil many classical intuitions
  - 2011 Nobel Prize in Chemistry
- Show up in many places
  - Material physics: built in labs, found in nature
  - CMT: lattice models, topological phases
  - Math: logic, non-commutative geometry
  - HEP: 2d CFT, discrete holography?





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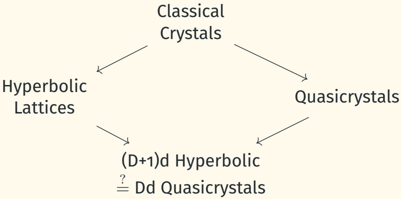

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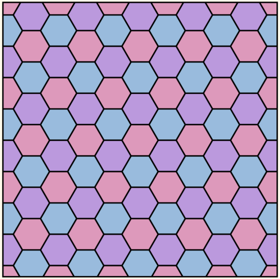
Beyond Euclidean Lattices

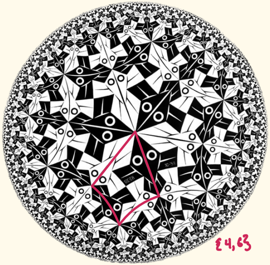
Regular Tilings in 2D



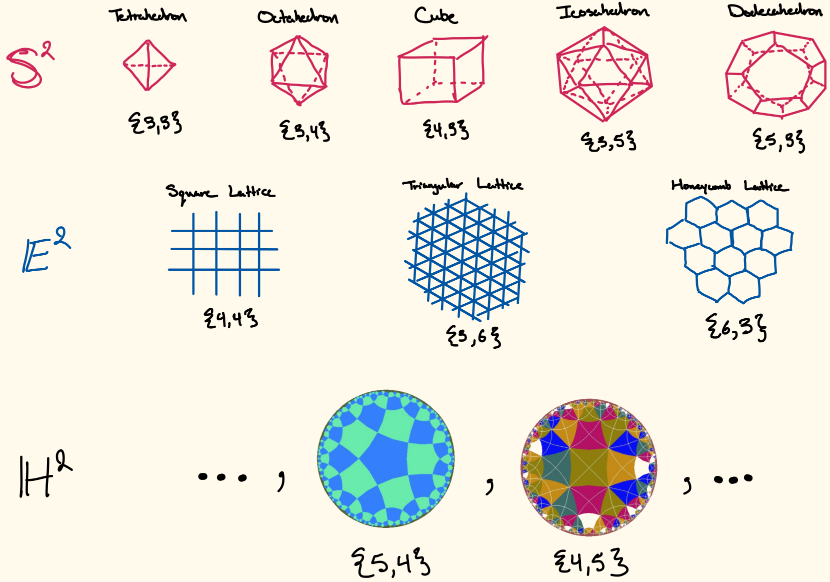
- In the Euclidean plane, the angle of a regular p-gon $\{p\}$ is $(1-2/p)\pi$
  - Regular tilings of $\mathbb{R}^2$: {3,6}, {6,3}, {4,4}
  - In higher dimensions, must make sure that dihedral angles fit around edges, etc.

- In the Hyperbolic plane, angles of a regular p-gon gradually decrease to zero as the edges get bigger.
  - We can use this to tune the p-gons to fit around a vertex

$$\frac{1}{p}+\frac{1}{q}\begin{cases} > \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{S}^2 \\ = \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{E}^2 \\ < \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{H}^2 \end{cases}.$$

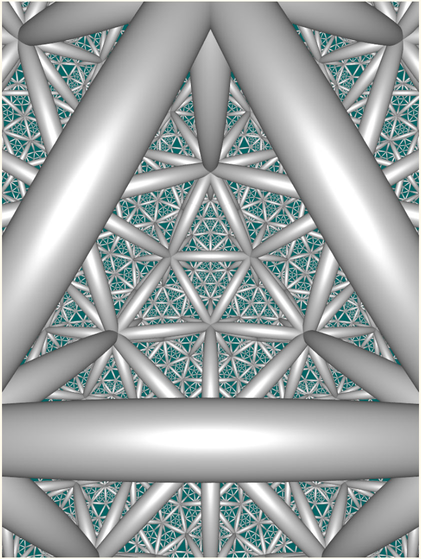


Regular Tilings in 2d: Pictures

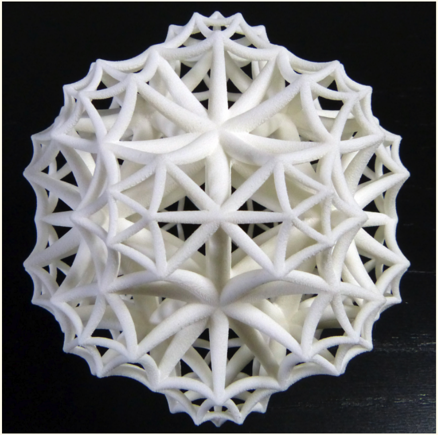


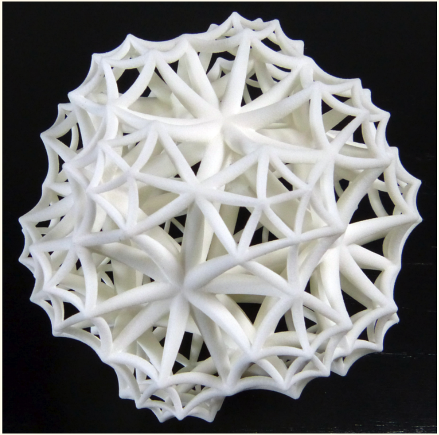
Regular Honeycombs in Hyperbolic Space

- $\mathbb{H}^2$ $\{p, q\}$ where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$
- $\mathbb{H}^3$ $\{4, 3, 5\}$, $\{5, 3, 4\}$, $\{5, 3, 5\}$, and $\{3, 5, 3\}$ [right]
- $\mathbb{H}^4$ $\{3, 3, 3, 5\}$, $\{5, 3, 3, 3\}$, $\{4, 3, 3, 5\}$, $\{5, 3, 3, 4\}$, and $\{5, 3, 3, 5\}$
- $\mathbb{H}^5$ Only some with non-finite cells and vertex figures.
- $\mathbb{H}^d$ When $d > 5$, none at all.

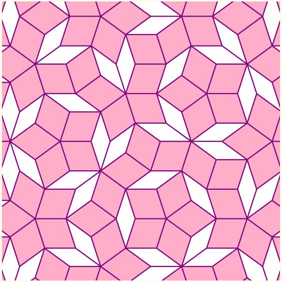


The {3,5,3} Hyperbolic Honeycomb Outside



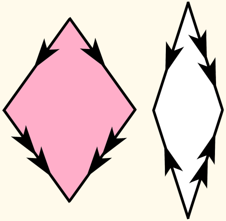


Quasicrystals

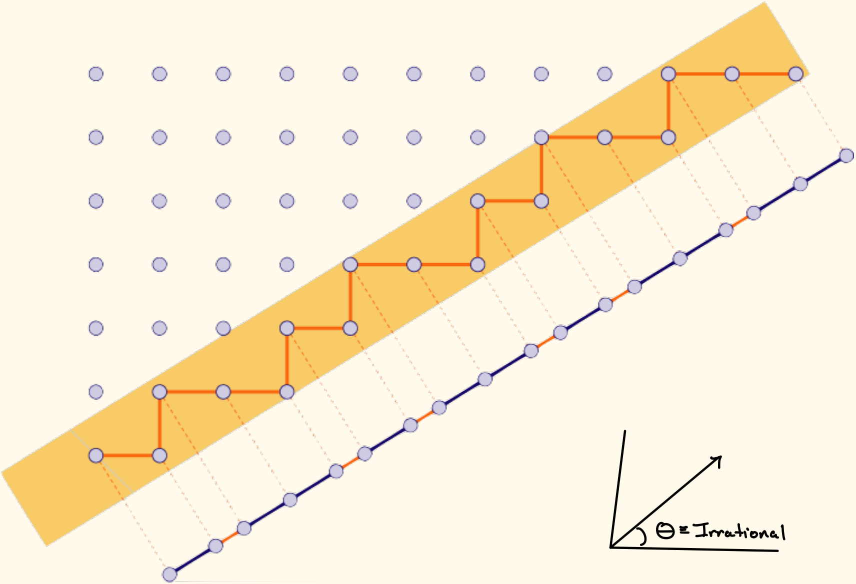


- **Quasicrystals** are aperiodic tilings which still possess order.
  - **Aperiodic** means no translation symmetry by any amount at all.
  - Define by their diffraction pattern: reciprocal lattice is spanned by $\mathbb{Z}$-linear combinations of $> d$ basis vectors in d dimensions.
- There are many ways to create a quasicrystal (not all of which are equivalent).
  - Constructions highlight different properties.

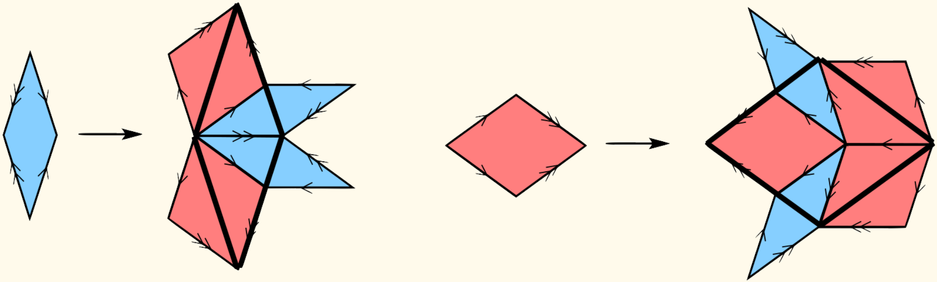
1. **Local Matching Rules**.
2. **Ammann Lines**. Long-range order
3. **Cut and Project**. Forbidden symmetries
4. **Substitution Rules**. Aperiodicity



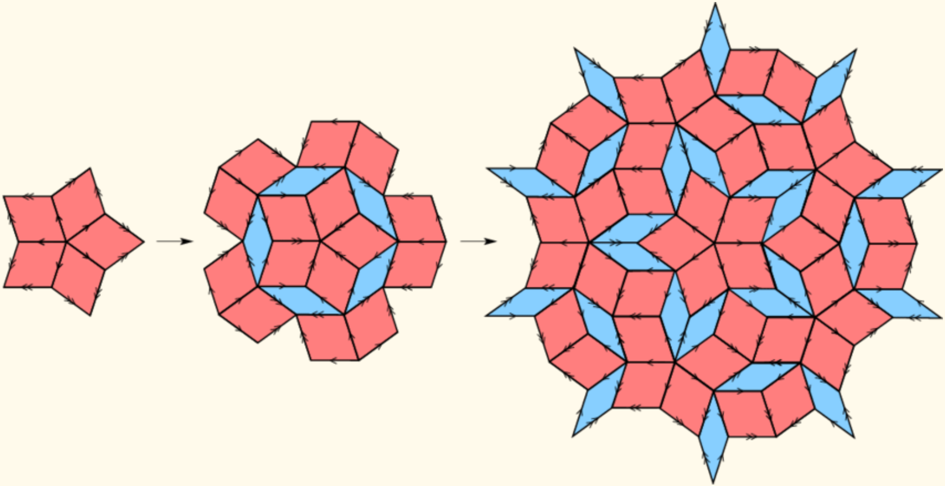
Cut and Project



Substitution Rules: Definition

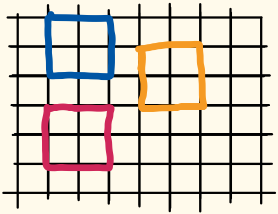


Substitution Rules: In Action



Substitution Rules: Implies Aperiodic

- Start with a finite set of (decorated) prototiles $\tau = \{T_1, \ldots, T_N\}$
- Suppose there is a unique rule for grouping prototiles into larger "super-tiles," with **scale factor** λ



- Then τ is an aperiodic set.

Ex. Penrose tiling has two prototiles; supertiles have $\lambda = \frac{1}{2}(1+\sqrt{5})$.

Ex. (Non-Example) There is no unique/local way to group an unmarked square tiling into larger squares. Domain walls can form.

- Capture "combinatorial information" in the **substitution matrix**.

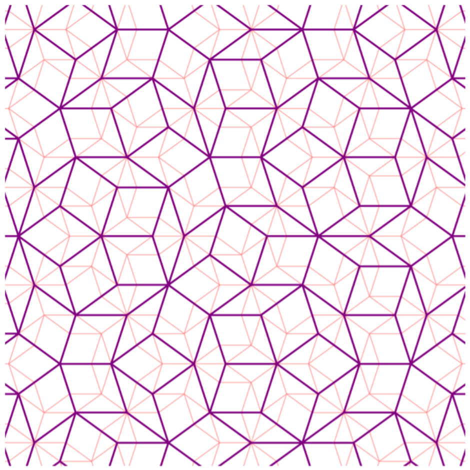
! For aperiodic tilings, dominant eigenvalue is $\lambda^2 > 1$ and irrational.

$$\begin{pmatrix} \text{Thin}' \\ \text{Thick}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \text{Thin} \\ \text{Thick} \end{pmatrix} \tag{2}$$

How many Penrose Tilings are there?

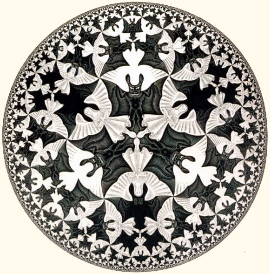
! There are an uncountable infinity of Penrose Tilings.

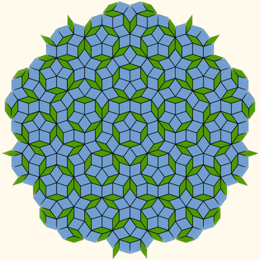
- Penrose tilings are **Locally Indistinguishable**: any finite-sized patch of tiles in one can be found in any other.
- Penrose tilings have **local scale symmetry**: take any tiling, decimate it, rescale it by λ. This new tiling is LID from the original (but globally distinct).



Summary of Background

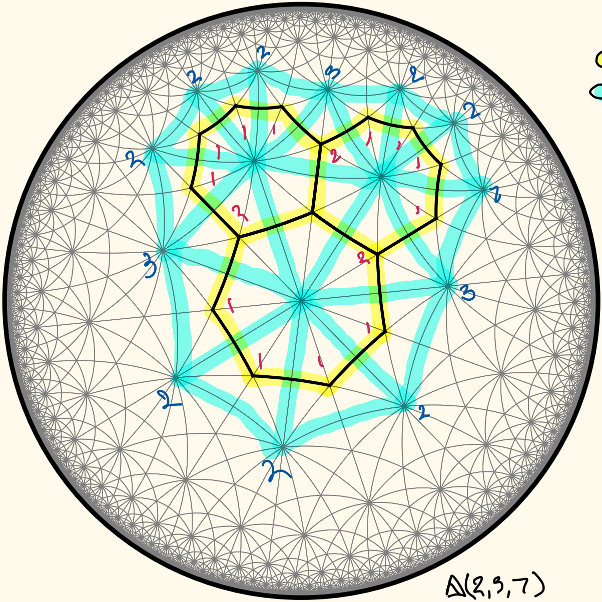
- **Hyperbolic space** exists:
  - You can tile it with **regular tessellations**. There are infinitely many $\{p, q\}$ tilings in 2d, and a few more in higher dimensions.
- **Quasicrystals** exist:
  - Aperiodic. Classically forbidden symmetries. Long-range order.
  - Many constructions; most useful for us is existence of a unique (invertible) local **substitution rule** with irrational scale factor λ.
  - Most famous example is the Penrose tiling.





Hyperbolic Quasicrystals 2d/1d

Growing Hyperbolic Honeycombs



(yellow) = $\{7,3\}$

(cyan) = $\{3,7\}$

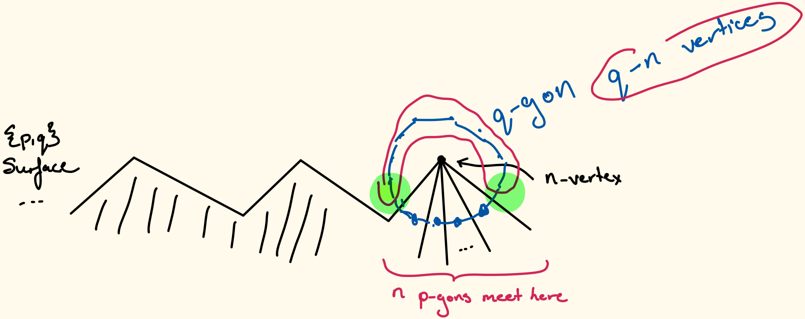
$1 \mapsto 2^{1/2}\, 2^{1/2}$

$2 \mapsto 2^{-1/2}\, 3\, 2^{-1/2}$

$2 \mapsto 2^{1/2}\, 1^3\, 2^{1/2}$

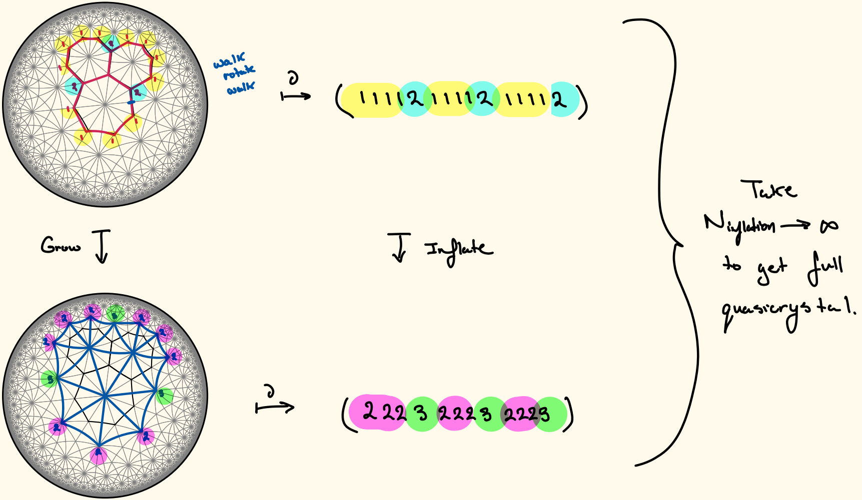
$3 \mapsto 2^{1/2}\, 1^2\, 2^{1/2}$

A General Rule in 2d/1d



$$n \mapsto 2^{1/2}\, 1^{q-n-2}\, 2^{1/2}$$

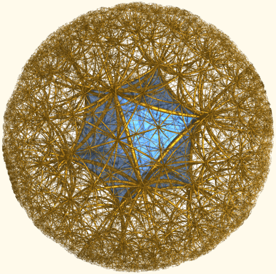
The Claim


walk
rotate
walk
∂
(11112 11112 11112)
Grow
Inflate
∂
(2223 2223 2223)
Take
N_inflation → ∞
to get full
quasicrystal.

Hyperbolic Quasicrystals in Higher Dimensions

Higher Dimensions and a Conjecture of Thurston

- A conceptual problem/test:
  - 1d is too topological: no curvature, where tiles live/lengths is pretty meaningless
  - Is this really a relationship between geometry of hyperbolic tessellations and quasicrystals?
- Can we identify a similar relationship in higher dimensions?
  - Recall in $\mathbb{H}^3$ there were 4 regular tessellations: $\{4,3,5\}$, $\{5,3,4\}$, $\{5,3,5\}$, and $\{3,5,3\}$ [right]

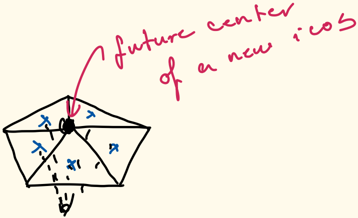


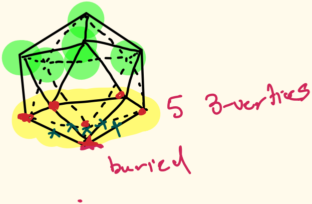
Conjecture: Boyle (2019) Thurston (?)

The boundary of the {3,5,3} lattice looks locally like the Penrose tiling.

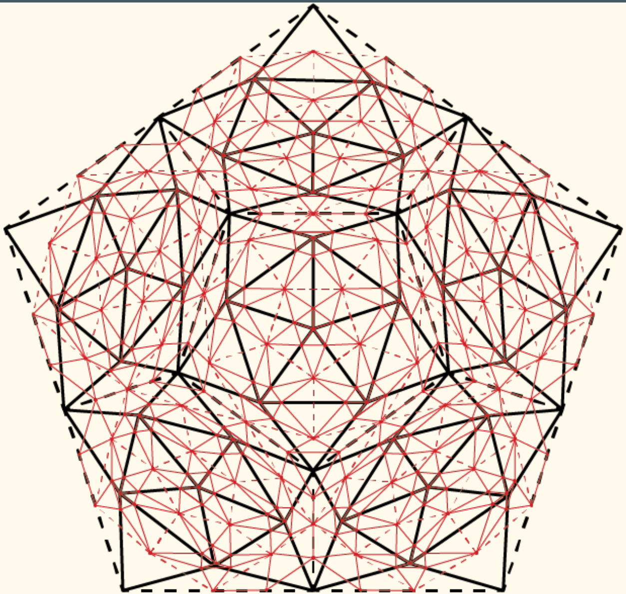
Growing {3,5,3}

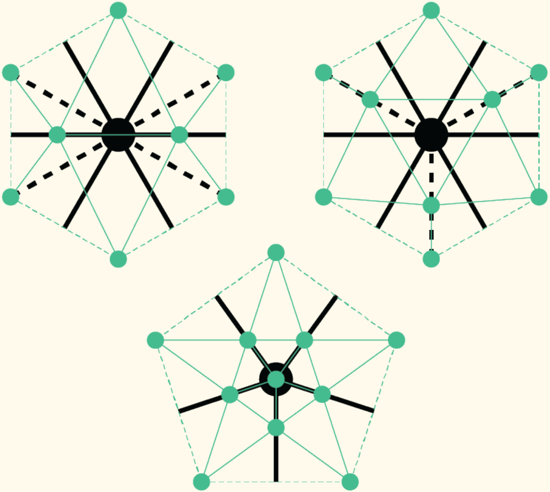
By carefully thinking about how the bulk 3d patch grows, and making a careful analogy to the 2d bulk examples, we will see how 2d boundary tile data is encoded in the species and relative configurations of boundary vertices.





{3,5,3} Projected





Resolution of Conjecture

- 3 natural tiles in the growth of a single icosahedral cell
  - With more complicated starting configurations, there will be more complicated tiles that can appear on the boundary
  - Procedure defines a **consistent 2d aperiodic substitution rule**
- Using further constraints (like χ) one can show (similar to 2d) that you only need to consider two prototiles (in principle)

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- Using further constraints (like χ) one can show (similar to 2d) that you only need to consider two prototiles (in principle)
- The resulting 3d/2d substitution rule turns out to have scale factor $\lambda_{\{3,5,3\}} = \lambda^2_{\text{Penrose}}$.
  - A priori, this doesn't mean our answer is wrong, perhaps we just didn't find a fine-grained enough set of boundary tiles.
  - By consider the space of Penrose tilings, can show there is **no refinement** that produces Penrose tilings.
  - **Thurston's conjecture** appears to be **false**.

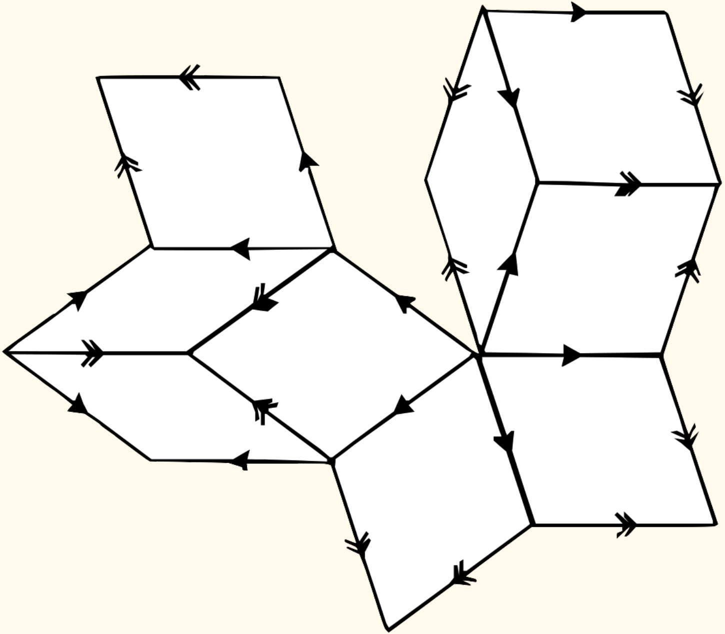
Conclusion

Outlines and Punchlines

1. Hyperbolic lattices and quasicrystals
   - Hyperbolic lattices and quasicrystals exist. There appears to be a relationship between them.
   - What does the topping-out of regular lattices in $\mathbb{H}^4$ mean?
2. New quasiperiodic patterns
   - 2d Quasicrystals with 5-fold symmetry are all the Penrose Tiling or close cousins: produced a **new 5-fold symmetric quasicrystal** from hyperbolic space techniques.
   - Is there a similar near-miss for the Elser-Sloane quasicrystal?
3. Applications to physics (holography, condensed matter)
   - Can we extend the results of Erdmenger et al to higher dimensions?

Fin

Failed Tiling



Ammann Lines