Twisted Tools for (Untwisted) Quantum Field Theory

University of Toronto - HEP Seminar

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Outlines and Punchlines 1/2

- Discuss ideas from formal deformation theory in QFT
  - Familiar example is Ocneanu rigidity of fusion categories
  - Ocneanu rigidity is ’t Hooft anomaly matching in QFT
- QFTs have “higher” multilinear ***k*-ary operations** (“brackets”)

$$\{-, -, \ldots, -\} \tag{1}$$

  - Control: **deformations**, (generalized) **OPEs**, and **anomalies**
  - Satisfy associativity relations, generalizing Jacobi identity
- Familiar to high energy physicists who have studied twisted SQFTs (descent relations)
  - Not limited to twisted scenarios
  - Can go very far in the case of HT theories
- Systematically compute the “semi-chiral” or $\frac{1}{4\mathcal{N}}$-BPS operators in supersymmetric QFTs

Outlines and Punchlines 2/2

I. Introduce the η-function and higher brackets
II. Compute the η-function and generalizations
III. Twisted SQFTs
IV. Cohomology and Confinement

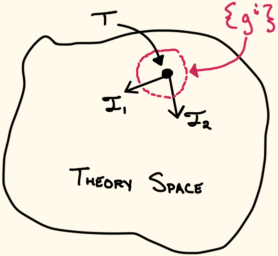
Three Takeaways

1. QFTs have higher brackets, defined by the η-function, which contain OPEs/anomalies/etc.
2. η-vector is very computable, especially in HT scenarios.
3. Systematic computation of semi-chiral ring + corrections

The η-Function

- Given a QFT T, it can be deformed by turning on interactions

$$S_T + \sum_i g^i \int_{\mathbb{R}^d} \mathcal{I}_i(x) d^d x \qquad (2)$$

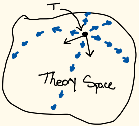


  - g^i are “coordinates” on “theory space”
  - Work perturbatively in couplings g^i
- Defines a **formal pointed neighbourhood** $\mathcal{D}[T]$ of T, of all effective QFTs obtained by perturbative deformation of T
  - Neighbourhood includes tangent space and all “higher tangent spaces” to the space of theories.
  - Tangent space to space of theories are interactions
  - Infinitesimally captures curvature of space of theories.

The Beta Function

- Generic QFT (point) is not scale invariant
  - Scale transformation on T is traded for a change of the couplings e.g. integrating out DoF modifies couplings
- We encode infinitesimal scale transformations in a vector field on theory space, the **beta function**

$$\boldsymbol{\beta} = \sum_i \beta^i(g) \frac{\partial}{\partial g^i} \tag{3}$$



  - Perturb around (typically free) scale-invariant theory, $\boldsymbol{\beta} = 0$
  - Deformations of T preserving scale invariance are zeroes of $\boldsymbol{\beta}$
- The coefficients $\beta^i(g)$ are power series in g

$$\beta^i(g) = \underbrace{(d - \Delta_i)}_{\text{Classical}} g^i + \underbrace{O(g^2)}_{\text{Quantum}} \tag{4}$$

  - Usually tune relevant terms to 0 and study $\boldsymbol{\beta}$ as a measure of scale generated by "quantum effects"

- Quantization of non-abelian gauge theories is hard; formulated redundantly in exchange for other properties

$$Z = \int [DAD\bar{\psi}D\psi]\, e^{-S[A,\bar{\psi},\psi]} =: \int [D\Phi]\, e^{-S[\Phi]} \tag{5}$$

  - Introduce gauge fixing procedure and FP ghosts b and c

$$Z = \int [D\Phi D B_A\, db_A\, dc^\alpha]\, e^{-S[\Phi] + iB_A F^A[\Phi] - b_A c^\alpha \delta_\alpha F^A[\Phi]} \tag{6}$$

- Gauge fixed action has nilpotent odd global symmetry involving fields and ghosts, called **BRST symmetry**.

$$\begin{aligned} \delta_{\mathrm{BRST}}\Phi &= -i\epsilon c^\alpha \delta_\alpha \Phi\,, & \delta_{\mathrm{BRST}} B_A &= 0\,, \\ \delta_{\mathrm{BRST}} c^\alpha &= \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma\,, & \delta_{\mathrm{BRST}} b_A &= \epsilon B_A\,. \end{aligned} \tag{7}$$

  - Physical theory can be identified with Q_{BRST}-cohomology

The BV-BRST Formalism 2/2

- Consider $T \hookrightarrow (\widetilde{T}, Q_{\mathrm{BRST}})$ described in a BV-BRST formalism.
  - $\widetilde{T}$ is a bigger ambient theory with ghosts, anti-ghosts, anti-fields, etc. and odd nilpotent symmetry
  - Observables in T are recovered from $\widetilde{T}$ by taking Q_{BRST} coho

$$\begin{aligned} \mathrm{Ops}_T &= (\mathrm{Ops}_{\widetilde{T}}, Q_{\mathrm{BRST}}) \\ \mathrm{Int}_T &= (\mathrm{Ops}_{\widetilde{T}}[dx], \mathrm{d} + Q_{\mathrm{BRST}}) \end{aligned} \tag{8}$$

- BV-BRST formalism is natural and necessary
  - Essential to quantizing p-form gauge theories, theories which only close on-shell, field-dependent structure constants...
  - Not restricted to such complicated theories either (e.g. scalar)
  - QFT in 0-dimensions is like studying integration on M; the BV formalism is like studying all of $\Omega^\bullet(M)$.
  - Building EFTs and accounting for EoM? **BV is Dyson-Schwinger**

The Eta Function

- Can compute analogue of β **for any type of transformation**.
  - Ex. Non-relativistic scale transformations $(t, x) \mapsto (\lambda^z t, \lambda x)$
  - Ex. Anomalous axial transformation on θ angle in gauge theory
- Consider $T \hookrightarrow (\widetilde{T}, Q_{\mathrm{BRST}})$ described in a BV-BRST formalism
  - Consider deformations of $\tilde{T}$ with Grassmann odd couplings, non-trivial ghost number, etc. This is a formal pointed dg-supermanifold $\mathcal{D}[\widetilde{T}]$.
  - To deform T, we deform $\widetilde{T}$ without breaking BRST symmetry.
- BRST symmetry will be encoded in a vector field

$$\boldsymbol{\eta} = \sum_i \eta^i(g) \frac{\partial}{\partial g^i} \tag{9}$$

  - Linear term tells us if $\mathcal{I}$ explicitly/classically violates BRST symmetry. Higher order terms do so "quantum mechanically"

- Since $Q_{\mathrm{BRST}}^2 = 0$, the eta function satisfies $\boldsymbol{\eta}^2 = 0$.
  - Wess-Zumino consistency condition for BRST symmetry
  - Gives **quadratic constraints** on coefficient functions $\eta^i(g)$

$$\eta^i(g) = \sum_{n>0} \frac{1}{n!} \sum_{j_1\cdots j_n} \eta^i_{j_1\cdots j_n} g^{j_1} \cdots g^{j_n} \tag{10}$$

- Define the following multilinear operation $\mathrm{Int}^{\otimes n} \to \mathrm{Int}$

$$\{\mathcal{I}_{j_1}, \ldots, \mathcal{I}_{j_n}\} = \eta^i_{j_1\cdots j_n} \mathcal{I}_i \tag{11}$$

  - The BRST variation is a **Maurer-Cartan equation**

$$\boldsymbol{\eta}\mathcal{I} = \{\mathcal{I}\} + \frac{1}{2!}\{\mathcal{I},\mathcal{I}\} + \frac{1}{3!}\{\mathcal{I},\mathcal{I},\mathcal{I}\} + \ldots \tag{12}$$

$\boldsymbol{\eta}^2 = 0 \quad \Leftrightarrow$ Coefficients $\eta^i_{j_1\cdots j_n}$ and brackets $\{\cdot, \ldots, \cdot\}$ define an L_∞**-algebra** structure on Int.

η-Function Calculations and Data

η-Function Calculation - General 1/2

- η is not just abstract fun, it is computable fun
- Correlation functions of $T + \mathcal{I}$ are correlation functions of T with exponential insertion:

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{T+\mathcal{I}} = \left\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{g \int d^d x \, \mathcal{I}} \right\rangle_T \tag{13}$$

  - At **linear order**, the BRST anomaly generated by $\mathcal{I}_i$ is just

$$\int_{\mathbb{R}^d} [Q, \mathcal{I}_i] \tag{14}$$

  - Write

$$[Q, \mathcal{I}_i] = \sum_j \eta_i^j \mathcal{I}_j + d\mathcal{J}_i \tag{15}$$

- In perturbation theory, higher order terms

$$O(g^n) \sim \int_{\mathbb{R}^{dn}} \mathcal{I}(y_1) \cdots \mathcal{I}(y_n) \tag{16}$$

  - Need **regularization** to avoid UV divergences from colliding $\mathcal{I}$

η-Function Calculation - General 2/2

Ex. **At** $O(g^2)$ we can regularize the deformation to

$$\int_{\mathbb{R}^{2d}} f_\epsilon^{(2)}(x_1, x_2)\mathcal{I}(x_1)\mathcal{I}(x_2) \tag{17}$$

- We can compute:

$$\begin{aligned}\Big[Q, \int_{\mathbb{R}^{2d}} f_\epsilon^{(2)}(x_1, x_2)\ \mathcal{I}(x_1)\,\mathcal{I}(x_2)\Big] \\ = -\int_{\mathbb{R}^{2d}} df_\epsilon^{(2)}(x_1, x_2)(\mathcal{I}(x_1)\mathcal{J}(x_2) + \mathcal{J}(x_1)\mathcal{I}(x_2))\end{aligned} \tag{18}$$

- With a sharp cutoff (**point-splitting**) this becomes

$$\{\mathcal{I}, \mathcal{I}\}(x_2) \overset{\text{Sharp Cutoff}}{=} \int_{|x_{12}|=\epsilon} \mathcal{I}(x_1)\mathcal{J}(x_2) + \mathcal{J}(x_1)\mathcal{I}(x_2) \tag{19}$$

- Quantum BRST anomaly appears from total derivative giving a boundary contribution in point-splitting regularization.

- 2d S_{Matter} with G global symmetry and G gauge theory

$$S_T = -\frac{1}{4}\int d^2x\, F_{\mu\nu}F^{\mu\nu} + S_{\text{Matter}} \tag{20}$$

Ex. Free fermions with vector current $J_a^\mu = \bar{\psi}\gamma^\mu t_a \psi$.

- Study the **interaction** $\mathcal{I} = A_\mu J^\mu$

- Add ghosts and auxiliary fields $T \hookrightarrow (\widetilde{T}, Q_{\text{BRST}})$
  - BRST transformation of $\mathcal{I}$ gives:

$$\delta_{\text{BRST}}(A_\mu J^\mu) = (\epsilon D_\mu c)J^\mu + A_\mu(i\epsilon g c J^\mu) = \epsilon \partial_\mu c J^\mu \tag{21}$$

  - See $\mathcal{I}$ is BRST-closed up to total derivative $\mathcal{J} = cJ$, which can potentially cause BRST anomaly

Basic Eta-Function Calculation - Concrete 2/2

- The two-bracket receives a contribution from the 2d JJ OPE:

$$\{\mathcal{I},\mathcal{I}\}(x_2) = \int_{|x_{12}|=\epsilon} :AJ:(x_1)\ :cJ:(x_2) + :cJ:(x_1)\ :AJ:(x_2)$$
$$= \oint_{S^1_{x_2}} \left(:A(x_1)c(x_2): + :c(x_1)A(x_2): \right) \langle J(x_1)J(x_2) \rangle$$
$$= \#\ :c\, dA:\ (x_2)\,. \tag{22}$$

  - We use $JJ \sim |x_{12}|^{-2}$, Taylor expanded, and integrated by parts
  - $\#$ denotes combinatorial and rep-theoretic factors

- Recover well-known **1-loop anomaly for G-gauge theory**

$$\{A_\mu J^\mu, A_\nu J^\nu\} = \#\, cF_{12}\,. \tag{23}$$

  - In $2k$-dim, you recover anomaly from $(k+1)$-ary bracket

Perturbative Corrections to Q

- **Callan-Symanzik equation** says renormalized correlators are independent of (arbitrary) renormalization scale μ:

$$\mu \frac{d}{d\mu} G^{(n)} = \left(\mu \frac{\partial}{\partial \mu} + \boldsymbol{\beta} + \boldsymbol{\gamma} \right) G^{(n)} = 0 \tag{24}$$

- Analogue of γ is $\boldsymbol{Q}$ on local operators
  - Coefficients $Q^i(g)$ of $\boldsymbol{Q}$ define operations $\mathrm{Int}^{\otimes n} \otimes \mathrm{Op} \to \mathrm{Op}$

$$\boldsymbol{Q}\,\mathcal{O} = \{\mathcal{O}\} + \{\mathcal{I}, \mathcal{O}\} + \frac{1}{2}\{\mathcal{I}, \mathcal{I}, \mathcal{O}\} + \cdots . \tag{25}$$

- Compute “semi-chiral ring” in interacting SQFTs

Q_{BRST} acts on fields and generates a vector field $\boldsymbol{\eta}$ on the space of theories/perturbative interactions. $\boldsymbol{Q}$ describes the **quantum loop corrected action** of BRST symmetry on **local operators**.

Midpoint Recap and Generalizations

- Recap so far:
  - Deformations are integral to our understanding of QFT
  - BV-BRST formalism introduces η-function, geometrizes QFT deformation theory by tracking BRST anomalies
  - η defines an L_∞-algebra on interactions and Maurer-Cartan equation is solved by well-defined deformations
  - η contains useful information like anomalies and OPEs
  - Can compute deformed/interacting $\mathbf{Q}$ on local operators
- Generalizations:
  - Can consider **position dependent interactions**: causes momentum-inflow p^i at each vertex. L_∞-brackets extend to

$$\otimes_i \mathrm{Int}_{p^{(i)}} \to \mathrm{Int}_{\sum_i p^{(i)}} \tag{26}$$

  - **Auxiliary and defect systems**: the brackets of $T \times T_{\text{probe}}$ extract more information about T

    Ex. All OPE coefficients of chiral VOA from auxiliary brackets

Twisted SQFTs

Why Supersymmetry?

- **SUSY QFTs** provide a rich, but tractable, collection of theories in which quantities are exactly computable or highly constrained; including **phases of gauge theories**
  - Non-renormalization theorems and dualities
- Insights from SUSY QFTs can tell us about the real world

  Ex. **Super QCD**. $\mathcal{N}=1$ $SU(N_c)$ SYM with N_f fundamentals exhibits theories with confinement, chiral symmetry breaking, strongly coupled IR CFTs etc.
- Tractability follows from existence of **protected quantities**
  - Can be invariant under deformation of coupling constant
  - Computable in different duality frames; probe NP physics

  Ex. Superconformal index counts local operators with signs

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{j_1+j_2-r/2} q^{j_1-j_2-r/2} e^{-\beta\{Q_-,S_+\}} \tag{27}$$

- **Supersymmetry** enhances Poincaré symmetry

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = P_{\alpha\dot\beta}\,, \tag{28}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\,. \tag{29}$$

  - Work with 4d $\mathcal{N} = 1$ (in Euclidean signature) for simplicity
- Pick some supercharge $Q = Q_-$
  - The **semi-chiral ring** of Q consists of all Q-invariant operators

$$[Q, \mathcal{O}] = 0 \tag{30}$$

  - In SUSY vacuum correlators of Q-invariant operators satisfy

$$\langle(\mathcal{O} + [Q, \Lambda])\cdots\rangle = \langle\mathcal{O}\cdots\rangle + \langle[Q, \Lambda\cdots]\rangle = \langle\mathcal{O}\cdots\rangle \tag{31}$$

  - So don't care about operators modulo $\mathcal{O} \sim \mathcal{O} + [Q, \Lambda]$.

From SUSY to the Holomorphic Twist

- Given any SQFT, we define the **Holomorphic Twist** by taking **cohomology** of any one nilpotent supercharge, e.g. $Q := Q_-$

$$Q^2 = 0\,, \quad Q\text{-Closed: } [Q, \mathcal{O}] = 0\,, \quad Q\text{-Exact: } [Q, \Lambda]\,, \tag{32}$$

  - Most available & least forgetful twist: only needs $\mathcal{N} = 1$ SUSY.
  - Cohomology isolates the semi-chiral ring
  - Cohomology classes computed by the holomorphic twist are the objects counted by the superconformal index.

- Anti-holomorphic translations are Q-exact, so twisted theory is (cohomologically) **holomorphic** [Johansen], [Nekrasov], [Costello]

$$\{Q, \bar{Q}_{\dot{\alpha}}\} = \partial_{\bar{z}^{\dot{\alpha}}} \tag{33}$$

  - $\mathrm{Spin}(4) \rightsquigarrow SU(2)$
  - We added Q_- to Q_{BRST} and trivialized anti-holo. translations

Superfields and Dolbeault Cohomology

- 4d $\mathcal{N} = 1$ SQFTs are formulated in language of superspace.
- We use a **chiral superspace**, with only $\bar{\theta}^{\dot{\alpha}}$ coordinates
  - A superfield is of the form

$$\mathcal{O}[\bar{\theta}] = e^{\bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}} \mathcal{O}^{(0)} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} \tag{34}$$

  - The right-handed supercharges act by $\bar{Q}_{\dot{\alpha}} = \partial_{\bar{\theta}^{\dot{\alpha}}}$
- Identify $\bar{\theta}^{\dot{\alpha}} \leftrightarrow d\bar{z}^{\dot{\alpha}}$, superfields are **Dolbeault $(0, \bullet)$-forms**
- Subsector of original QFT encoding $1/4\,\mathcal{N}$-BPS operators and their SUSY multiplets in holomorphic fields on spacetime
  - Twisted stress tensor $S_{\dot{\alpha}}$ is a chiral piece of $\mathcal{S}$-multiplet
  - (Derivative of) $S_{\dot{\alpha}}$ generates remaining $SU(2)$ symmetries
  - Partition function is superconformal index

A 2d CFT Reminder

- 2d CFTs have infinite dimensional symmetry enhancements
  - Every holomorphic local operator is automatically $\bar{\partial}$-conserved and remains so when multipled by some $f(z)$.
  - Gives humongous families of symmetries, e.g. Virasoro
- A **2d chiral conformal primary** ϕ has a mode expansion:

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\hat{\phi}_n = z^{-h}\left(\cdots + z^{-1}\hat{\phi}_1 + z^0\hat{\phi}_0 + z^1\hat{\phi}_{-1} + \dots\right)$$

  - To **extract the mode $\hat{\phi}_n$**, complex analysis says that:

$$\hat{\phi}_n = \oint \frac{dz}{2\pi i} z^{n+h-1}\phi(z) \tag{35}$$

  - View this as wedging the $(0,0)$-form $z^h\phi(z)$ with a form:

$$\phi(z) \mapsto \frac{dz}{2\pi i} z^{n-1} \wedge z^h\phi(z) \tag{36}$$

- 2d chiral primaries have a $\mathbb{Z}$ of modes "because"

$$H^{1,\bullet}(\mathbb{C}^1\setminus\{0\}) \cong \mathbb{Z} \tag{37}$$

Modes in the Holomorphic Twist

- On $\mathbb{C}^2$, chiral superfields $\mathcal{O}$ are $(0,\bullet)$-forms
- For any $\rho \in H^{2,\bullet}(\mathbb{C}^2 \setminus \{0\})$ we define

$$\hat{\mathcal{O}}_\rho = \oint_{S^3} \mathcal{O} \wedge \rho \tag{38}$$

  - Deg 0 $\rho \sim z_1^n z_2^m d^2 z$ gives **non-negative modes** of operator
  - Deg 1 $\rho \sim \partial^n_{z^1} \partial^m_{z^2} \omega_{\mathrm{BM}}$ gives **negative modes** of operator
- **Infinite dimensional symmetry enhancements** analogous to Virasoro and Kac-Moody [Gwilliam, Williams].
  - Deformation $\rightsquigarrow$ [Beem, Lemos, Liendo, Peelaers, Raselli, van Rees]
  - [Bomans, Wu] central extensions of higher Virasoro exist, and are labelled by conformal anomalies (a and c), obtainable from (higher!) brackets of holomorphic stress-tensor
- Stress: OPE of a subsector of the **original physical theory**, *not* a deformed/modified theory.

Cohomology and Confinement

Free Cohomology and Adding Interactions

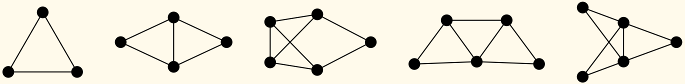
- Start with the **Free Cohomology** $\mathcal{V}$, i.e. free semi-chiral ring
  - G-inv. polynomials (words) in fields and derivatives (letters)
- One way to think of twisted theory is that we have added Q_{SUSY} to Q_{BRST} and trivialized some translations
  - **Interacting quantum Q** is obtained by computing brackets:

$$\mathbf{Q}\,\mathcal{O} = \{\mathcal{I}, \mathcal{O}\} + \frac{1}{2}\{\mathcal{I}, \mathcal{I}, \mathcal{O}\} + \frac{1}{6}\{\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{O}\} + \ldots\,. \quad (39)$$

  - We thus compute **all perturbative corrections** to the semi-chiral ring by computing the (higher) brackets
  - Brackets are a generalized Konishi-Anomaly for $\boldsymbol{Q}$.
- Use to systematically construct cohomology of $\frac{1}{16}$-BPS operators in $\mathcal{N} = 4$ SYM [Chang, Lin], [Choi, Kim, Lee, Lee, Park]
- Cohomology is not fully-protected like index, but can still compute corrections systematically; **categorifies** the index

Feynman Diagrams

- Feynman diagrams are **Laman graphs**. [Budzik, Gaiotto, JK, Wu, Yu]



  - Arbitrary holomorphic-topological twists in arbitrary dimensions are captured by generalized-Laman graphs [Kontsevich], [Gaiotto, Moore, Witten]

- Arbitrary integral takes the form:

$$\mathcal{I}_\Gamma[\lambda; z] \equiv \int_{\mathbb{R}^{4|\Gamma_0|-4}} \bar{\partial} \left[\prod_{e \in \Gamma_1} \mathcal{P}(x_{e_0} - x_{e_1} + z_e, \bar{x}_{e_0} - \bar{x}_{e_1}) \right] \left[\prod_{v \in \Gamma_0'} e^{\lambda_v \cdot x_v} d^2 x_v \right] \tag{40}$$

  - Change of variables maps integral to Fourier transform of a polytope in loop momenta, the **operatope**.
  - Feynman integrals satisfy infinite collection of geometric **quadratic identities**; enforcing associativity
  - Can **bootstrap** all Feynman integrals from these identities.

Twisting $\mathcal{N} = 1$ SYM

- $\mathcal{N} = 1$ **SYM** is $SU(N)$ gauge theory with an adjoint fermion

$$\mathcal{L} = \tau \int d^2\theta \; \mathrm{tr}\, W_\alpha W^\alpha + \mathrm{c.c.} \tag{41}$$

- Twist is identified (in a non-trivial way) with a **holomorphic** bc **system** [Costello], [Elliot, Safronov, Williams], [Saberi, Williams]
  - Fields are collected in adjoint bosonic superfield b and (co)adjoint fermionic superfield c.
- The Lagrangian of this theory is

$$\mathcal{L}_{\text{twisted}} = \mathrm{Tr}\, b\left(\bar{\partial} c - \frac{1}{2}[c, c]\right) + \tau\, \mathrm{Tr}\, \partial_\alpha c\, \partial^\alpha c\,. \tag{42}$$

  - Free cohomology

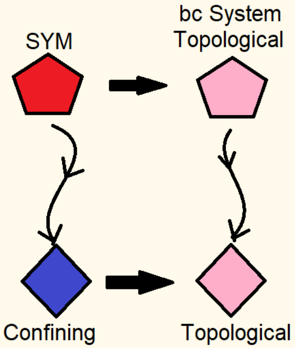
$$\mathbb{C}[b, \partial_\alpha b, \partial_\alpha \partial_\beta b, \ldots, \partial_\alpha c, \partial_\alpha \partial_\beta c, \ldots]^G \tag{43}$$

  - Derivative of the stress tensor is $\partial_\alpha S^\alpha = \partial_\alpha b_A \partial^\alpha c^A$.

Holomorphic Confinement

- Adding **one loop corrections**, we find $\partial_\alpha S^\alpha = Q \,\mathrm{Tr}\, b^2$
  - $\partial_\alpha S^\alpha$ generates all remaining spacetime symmetries.
  - Exactness of $\partial_\alpha S^\alpha$ means local operators are invariant under remaining spacetime transformations.
  - Theory becomes **topological** at one loop!

- Being topological is compatible with **confinement**: if topological in the UV, then topological in the IR.
  - Constrains IR physics: the holomorphic twist of the IR must also be topological.
  - **Holomorphic Confinement**



Final Recap

RECAP

Three Takeaways

1. QFTs have higher brackets, defined by the η-function, which contain OPEs/anomalies/etc.
   - η defines L_∞ algebra on space of interactions
   - η brackets characterize violation of BRST symmetry
2. η-vector is very computable, especially in HT scenarios.
   - All graphs are Laman graphs
   - Graphs satisfy "operatope" identities, which enforce associativity (or $Q^2_{\mathrm{BRST}} = 0$) diagram-by-diagram.
3. Holomorphic twist is local QFT of $\frac{1}{4\mathcal{N}}$-BPS operators
   - Twisted theory has infinite dimensional symmetry enhancements
   - Systematic computation of semi-chiral ring + corrections
   - Holomorphic confinement of $\mathcal{N} = 1$ SYM

Fin

Selected Historical SUSY References

- SUSY Non-Renormalization Theorems
  - [Sohnius, West], [Mandelstam], [Grisaru, Rocek, Siegel], [Seiberg], [Argyres, Plesser, Seiberg]
- Phases of gauge theories and SUSY breaking
  - Seiberg, Intrilligator, Strassler, Dine, Yu
- Supersymmetric dualities
  - [Montonen, Olive], Seiberg, Intrilligator, Witten, Argyres
- Superconformal index
  - [Witten], [Alvarez-Gaume], [Kinney, Maldacena, Minwalla, Suvrat], [Romelsberger], [Dolan, Osborn]
- Twisted SQFTs
  - Witten, Johansen, Donaldson

Holomorphic-Topological Theories

Holomorphic-Topological Theories

- “**Holomorphic-Topological**” means flat spacetime has structure of $\mathbb{C}^H \times \mathbb{R}^T$ with coordinates $(x^{\mathbb{C}}, \bar{x}^{\mathbb{C}}, x^{\mathbb{R}})$
  - Anti-holomorphic translations in $\mathbb{C}^H$ and translations in $\mathbb{R}^T$ are gauge symmetries (Q_{BRST}-exact)
  - Just like superspace lets us build intrinsically supersymmetry invariant actions, we can build BV actions with “superfields” Φ where $dx^{\mathbb{R}}$ and $d\bar{x}^{\mathbb{C}}$ are treated as “superspace coordinates.”
- Interested in theories with action

$$\int_{\mathbb{C}^H \times \mathbb{R}^T} \left[(\Phi, \mathrm{d}\,\Phi) + \mathcal{I}(\Phi)\right]\; d^H x^{\mathbb{C}} \tag{44}$$

  - In free theory, BRST closed superfields satisfy descent:

$$Q\,\mathcal{O} + \mathrm{d}\,\mathcal{O} = 0 \tag{45}$$

  - Interaction $\mathcal{I}(\Phi)$ is BRST-closed up to total derivative.
- Appears in holomorphic-topological twists of SUSY theories.

Holomorphic-Topological Integrals

- In such theories, we will be interested in brackets of the form

$$\{\mathcal{O}_1 \,{}_{\lambda_1} \ldots {}_{\lambda_{n-1}}\, \mathcal{O}_n\} \tag{46}$$

  - Conjecturally all information of perturbative HT factorization algebras. See [Wang, Williams] for rigourous discussion.

- The **Feynman integrals** that contribute will take the form:

$$I_\Gamma(\lambda; z) = \int_{\mathcal{M}^{|\Gamma_0|-1}} \left[\prod_{v\in\Gamma_0}^{v\neq v_*} \mathrm{dVol}_v e^{\lambda_v \cdot x_v^{\mathbb{C}}}\right] \mathrm{d}\left[\prod_{e\in\Gamma_1} P_\epsilon(x_{e(0)} - x_{e(1)} + z_e)\right]$$

  - Schwinger parameterization recasts integral as Fourier xform of polytope in space of loop momenta, the **operatope**.
  - Operatope makes UV and IR finiteness of Feynman integrals manifest.

n-Laman Constraints

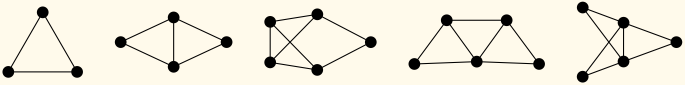
- Non-vanishing Feynman diagrams are **n-Laman graphs**
  - **Global Constraint**

$$n|\Gamma_0| = (n-1)|\Gamma_1| + n + 1 \tag{47}$$

  - **Local Constraint** For subgraphs $\Gamma[S]$

$$n|\Gamma[S]_0| \geq (n-1)|\Gamma[S]_1| + n + 1 \tag{48}$$

  - $n = H + T$ [Kontsevich], [Gaiotto, Moore, Witten], [Wang]



Quadratic Identities

- $I_\Gamma(\lambda; z)$ has a number of symmetries/identities: symmetries from the graph, and symmetries under shifts of z_e.
- Feynman integrals (generally diagrams) satisfy infinite collections of (geometric) **quadratic identities**:

$$\sum_{\text{Laman } S} \sigma(\Gamma, S) I_{\Gamma[S]}(\lambda + \partial; z) \cdot I_{\Gamma(S)}(\lambda; z) = 0\,. \tag{49}$$

  - Identities imply (higher)-associativity of the accompanying brackets in a diagram-by-diagram way
  - Can **bootstrap** all Feynman integrals from these identities?

Non-Renormalization Theorem

All loop graphs in $(H, T \geq 2)$-theories must vanish.

A Non-Renormalization Theorem

A Non-Renormalization Theorem 1/2

- Many interesting scenarios with mixed HT degree.

  Ex. (1+1)d holomorphic boundary of a (2+1)d TFT

  Useful to "forget" some of the structure of the bulk TFT.

- Trade $T = 2$ top. directions for an $H = 1$ holo. direction
  - Topological superfield Φ splits into two fields in the holomorphic theory, a $(0, *)$-form and a $(1, *)$-part:

$$\Phi = \Phi^{(0)} + \Phi^{(1)} dz \,. \tag{50}$$

  - Topological superfield condition also splits

$$(Q + \mathrm{d}_{\mathrm{top}})\Phi = (Q + \mathrm{d}_{\mathrm{Holo}})\Phi + \partial\Phi = 0 \,. \tag{51}$$

- $\partial\Phi$ term is now interpreted as a BRST anomaly due to the holomorphic part of the kinetic term
  - i.e. all top. theory calculations are replaced by calculations in identical holo. theory with extra $(\Phi, \partial\Phi)$ interaction

A Non-Renormalization Theorem 2/2

- Consider a calculation in $(H, T \geq 2)$
  - Convert to an equivalent calculation in $(H+1, T-2)$-theory

$$\begin{array}{ccc} (H, T \geq 2)\text{-theory} & \rightsquigarrow & (H+1, T-2)\text{-theory} \\ \Gamma & & \{\gamma_1, \gamma_2, \dots\} \end{array} \tag{52}$$

  - γ_i will correspond Γ with edges $e_i \in \Gamma_1$ broken into chains of edges $\{f_{i,1}, \dots, f_{i,m}\}$ in all possible ways
  - Each vertex has “two-point interaction” $(\Phi, \partial\Phi)$
  - Each γ_i has $|\Gamma_1| + k$ edges and $|\Gamma_0| + k$ vertices for some $k \geq 0$.
- If γ_i are non-vanishing, they must be $(n-1)$-Laman graphs.
  - Putting the two conditions together, Γ must be a tree for non-vanishing contribution

Infinite Dimensional Symmetry Cohomology

A 2D CFT REMINDER

- If ϕ is a **2d chiral conformal primary** with chiral dimension h, then we can expand

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\hat{\phi}_n = z^{-h}\left(\dots + z^{-1}\hat{\phi}_1 + z^0\hat{\phi}_0 + z^1\hat{\phi}_{-1} + \dots\right)$$

- If we want to **extract the mode $\hat{\phi}_n$**, we know that

$$\hat{\phi}_n = \oint \frac{dz}{2\pi i} z^{n+h-1}\phi(z) \tag{53}$$

  - Formally, extract $n \geq 0$ modes by multiplying by generator z^n for the space of holomorphic functions on $\mathbb{C}$
  - Extract $n < 0$ modes by integrating against Bochner-Martinelli kernel

$$\hat{\phi}_n = \frac{1}{n!}\phi^{(n)}(0) \propto \oint \partial^n \omega_{\mathrm{BM}}\phi(z) \tag{54}$$

    where $\omega_{\mathrm{BM}} = 1/z$

- These are elements of $H^{1,0}(\mathbb{C}^1\backslash\{0\})$. Note: $H^{1,1}(\mathbb{C}^1\backslash\{0\}) = 0$

DOLBEAULT COHOMOLOGY OF $H^{n,\bullet}(\mathbb{C}^n \setminus \{0\})$

- In other words

$$\hat{\phi}_n = \int_{S^1} [z^h \phi(z) d\bar{z}] \wedge \rho \,, \quad \rho \sim \begin{cases} z^n dz \\ \partial^n \omega_{\text{BM}} \end{cases} \in H^{1,0}(\mathbb{C} \setminus \{0\}) \,. \tag{55}$$

- $H^{2,\bullet}(\mathbb{C}^2 \setminus \{0\})$ is **concentrated in degrees 0 and 1**
  - Degree 0. Classes are (2,0)-Dolbeault forms with $n, m \geq 0$:

$$\rho \sim z_1^n z_2^m d^2 z \tag{56}$$

  - Degree 1. Classes are (2,1)-Dolbeault forms with $n, m \geq 0$:

$$\rho \sim \partial_{z^1}^n \partial_{z^2}^m \omega_{\text{BM}} \tag{57}$$

  - The **Bochner-Martinelli Kernel** is now the thing such that

$$\int_{S^3} \omega_{BM} f(z) = f(0) \tag{58}$$

    for any holomorphic $f(z)$ on $\mathbb{C}^2$.

Modes in the Holomorphic Twist

- In $\mathbb{C}^2$, we've seen that (chiral) superfields $\mathcal{O}$ are identifiable with $(0,\bullet)$-forms
- For any $\rho \in H^{2,\bullet}(\mathbb{C}^2 \setminus \{0\})$ we define

$$\hat{O}_\rho = \oint_{S^3} O \wedge \rho \tag{59}$$

  - Degree 0 classes $\rho \sim z_1^n z_2^m d^2 z$ give analogue of **non-negative modes** of VOA, $\hat{O}_{n,m} = \hat{O}_\rho$
  - Degree 1 classes $\rho \sim \partial_{z^1}^n \partial_{z^2}^m \omega_{\mathrm{BM}}$ give analogue of **negative modes** of VOA $\hat{O}_{-n-1,-m-1} = \hat{O}_\rho$
- In general $H^{n,\bullet}(\mathbb{C}^n \setminus \{0\})$ is ocncentrated in degree 0 (functions) and degree $n-1$ (dual functions)

Summary of Symmetries

Supercurrent Multiplet $S_{\alpha,\mu}$ $T_{\mu\nu}$

I define

$$S^{(0)}_{\dot\alpha} \equiv S_{+,+\dot\alpha}$$

It turns out

$$Q(\partial^{\dot\alpha} S^{(0)}_{\dot\alpha}) = 0$$

$\Leftrightarrow$ $\partial^{\dot\alpha} S_{\dot\alpha}$ is semi-chiral

Actually, when cons. R-sym

$S_{\dot\alpha}$ is semi-chiral

As it turns out, $S^{(1)}_{\dot\alpha}$ containing the holomorphic part of physical stress tensor and generates diffs of spacetime.

$\{\text{Negative } \hat{S}_{n,m}\}$

$\oplus$

Complex Symplectomorphisms i.e. Pres d^2z (Gen. by $\underbrace{\hat{S}_{n,m}}_{+ve}$)	$\subset$	Complex Diffeomorphisms (Needs R sym)
$\cup$		$\cup$
Global $\mathfrak{su}(2)$ subalgebra (Gen. by holomorphic part of $T^{\text{physical}}_{\mu\nu}$)	$\subset$	Global $\mathfrak{u}(2)$ subalgebra (Needs R sym)

Higher Virasoro

$SU(2)$ $\longrightarrow$ $U(2)$