Higher Brackets, BRST Anomalies, and Non-Renormalization

SQFTs, VOAs, and Geometry

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Outlines and Punchlines 1/2

- Discuss ideas from formal deformation theory in QFT
  - Familiar example is Ocneanu rigidity of fusion categories
- QFTs have "higher" multilinear **k-ary operations** ("brackets")

$$\{-,-,\ldots,-\} \tag{1}$$

  - Control: **deformations**, (generalized) **OPEs**, and **anomalies**
  - Satisfy associativity relations, generalizing Jacobi identity
- Familiar to high energy physicists who have studied twisted SQFTs (related to descent relations)
  - Can recover Moyal product, λ-brackets, higher n-Lie conformal algebras, etc. as special cases.
- Can be used to prove **non-renormalization theorem** in theories with $T \geq 2$ topological directions

I. Introduce the η-function and higher brackets
II. Describe this technology in HT theories ($\beta\gamma$-system)
III. HT integrals and non-renormalization theorem

Three Takeaways

1. QFTs have higher brackets, defined by the η-function, which contain OPEs/anomalies/etc.
2. η-vector is very computable, especially in HT scenarios.
3. Formalism can be used to compute and prove new things.

The η-Function

Deformations and the Beta Function

- Given a QFT T, it can be deformed by turning on interactions

$$S_T + \sum_i g^i \int_{\mathbb{R}^d} \mathcal{I}_i(x)\, d^d x \qquad (2)$$

  - g^i are "coordinates" on "theory space"
  - Work with formal power series in couplings g^i
  - Defines **formal pointed neighbourhood** $\mathcal{D}[T]$ of T, of all effective QFTs obtained by formal deformation of T

- Generic QFT (point) is not scale invariant
  - Scale xform on T is traded for a change of the couplings

- Encode infinitesimal scale transformations in **beta function**

$$\boldsymbol{\beta} = \sum_i \beta^i(g) \frac{\partial}{\partial g^i}\,, \qquad \beta^i(g) = \underbrace{(d - \Delta_i)}_{\text{Classical}}\, g^i + \underbrace{O(g^2)}_{\text{Quantum}} \qquad (3)$$

The BV-BRST Formalism

- Work in a BV-BRST formalism. i.e. embed $T \hookrightarrow (\widetilde{T}, Q_{\mathrm{BRST}})$
  - $\widetilde{T}$ is a bigger ambient theory with ghosts, anti-ghosts, anti-fields, etc. and odd nilpotent symmetry Q_{BRST}
  - Observables in T are recovered from $\widetilde{T}$ by taking Q_{BRST} coho

$$\begin{aligned} \mathrm{Ops}_T &= (\mathrm{Ops}_{\widetilde{T}}, Q_{\mathrm{BRST}}) \\ \mathrm{Int}_T &= (\mathrm{Ops}_{\widetilde{T}}[dx], \mathrm{d} + Q_{\mathrm{BRST}}) \end{aligned} \tag{4}$$

Ex. Quantization of non-abelian gauge theory:

$$Z = \int [DAD\bar{\psi}D\psi]e^{-S[A,\bar{\psi},\psi]} =: \int [D\Phi]e^{-S[\Phi]} \tag{5}$$

- Introduce gauge fixing procedure and FP ghosts b and c

$$Z = \int [D\Phi DB_A db_A dc^\alpha]e^{-S[\Phi]+iB_AF^A[\Phi]-b_Ac^\alpha\delta_\alpha F^A[\Phi]} \tag{6}$$

- Gauge-fixed action has nilpotent odd **BRST symmetry**.

$$\begin{aligned} \delta_{\mathrm{BRST}}\Phi &= -i\epsilon c^\alpha \delta_\alpha \Phi\,, & \delta_{\mathrm{BRST}}B_A &= 0\,, \\ \delta_{\mathrm{BRST}}c^\alpha &= \frac{i}{2}\epsilon f^\alpha_{\beta\gamma}c^\beta c^\gamma\,, & \delta_{\mathrm{BRST}}b_A &= \epsilon B_A\,. \end{aligned} \tag{7}$$

The Eta Function

- Can compute analogue of β **for any type of transformation**.

  Ex. Non-relativistic scale transformations $(t, x) \mapsto (\lambda^z t, \lambda x)$

- Consider $T \hookrightarrow (\widetilde{T}, Q_{\mathrm{BRST}})$ described in a BV-BRST formalism
  - Consider deformations of $\tilde{T}$ with Grassmann odd couplings, non-trivial ghost number, etc. This is a formal pointed dg-supermanifold $\mathcal{D}[\widetilde{T}]$.
  - To deform T, we deform $\widetilde{T}$ without breaking BRST symmetry.

- BRST symmetry will be encoded in **eta function**

$$\boldsymbol{\eta} = \sum_i \eta^i(g) \frac{\partial}{\partial g^i} \tag{8}$$

  - Linear term tells us if interaction $\mathcal{I}$ explicitly violates BRST symmetry. Higher order terms do so "quantum mechnically"

- Expand the $\boldsymbol{\eta}$-function coefficients $\eta^i(g)$

$$\eta^i(g) = \sum_{n>0} \frac{1}{n!} \sum_{j_1\cdots j_n} \eta^i_{j_1\cdots j_n} g^{j_1}\cdots g^{j_n} \tag{9}$$

  - $\eta^i_{j_1\cdots j_n}$ are structure constants for multilinear map $\mathrm{Int}^{\otimes n} \to \mathrm{Int}$

$$\{\mathcal{I}_{j_1},\ldots,\mathcal{I}_{j_n}\} = \eta^i_{j_1\cdots j_n}\mathcal{I}_i \tag{10}$$

- Since $Q^2_{\mathrm{BRST}} = 0$, the eta function satisfies $\boldsymbol{\eta}^2 = 0$.
  - **Quadratic constraints** on coefficients $\eta^i(g)$ and thus brackets
  - The BRST variation is a **Maurer-Cartan equation**

$$\boldsymbol{\eta}\mathcal{I} = \{\mathcal{I}\} + \frac{1}{2!}\{\mathcal{I},\mathcal{I}\} + \frac{1}{3!}\{\mathcal{I},\mathcal{I},\mathcal{I}\} + \ldots \tag{11}$$

$\boldsymbol{\eta}^2 = 0 \quad \Leftrightarrow$ Coefficients $\eta^i_{j_1\cdots j_n}$ and brackets $\{\cdot,\ldots,\cdot\}$ define an L_∞**-algebra** structure on Int.

η-Function Formulas

- Correlators of $T + g\mathcal{I}$ are correlators of T with insertion:

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{T+\mathcal{I}} = \left\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{g\int d^dx\,\mathcal{I}}\right\rangle_T \quad (12)$$

  - At **linear order**, the BRST anomaly generated by $\mathcal{I}_i$ is just

$$\int_{\mathbb{R}^d}\{\mathcal{I}_i\} = \int_{\mathbb{R}^d}[Q,\mathcal{I}_i] \quad (13)$$

  - Write $[Q,\mathcal{I}_i] = \sum_j \eta_i^j \mathcal{I}_j + d\mathcal{J}_i$

- In PT, $O(g^n)$-terms need **regularization** of UV divergences

$$O(g^2) \sim \int_{\mathbb{R}^{2d}} \mathcal{I}(x_1)\mathcal{I}(x_2) \mapsto \int_{\mathbb{R}^{2d}} f_\epsilon^{(2)}(x_1,x_2)\mathcal{I}(x_1)\mathcal{I}(x_2) \quad (14)$$

  - With a sharp-cutoff point-splitting regularization

$$\{\mathcal{I},\mathcal{I}\}(x_2) \overset{\text{Sharp Cutoff}}{=} \int_{|x_{12}|=\epsilon} \mathcal{I}(x_1)\mathcal{J}(x_2) + \mathcal{J}(x_1)\mathcal{I}(x_2) \quad (15)$$

EXAMPLE: 2D G GAUGE THEORY 1/2

- 2d S_{Matter} with G global symmetry and G gauge theory

$$S_T = -\frac{1}{4}\int d^2x\, F_{\mu\nu}F^{\mu\nu} + S_{\text{Matter}} \tag{16}$$

Ex. Free fermions with vector current $J_a^\mu = \bar{\psi}\gamma^\mu t_a \psi$.

- ▶ Study the **interaction** $\mathcal{I} = A_\mu J^\mu$

- Add ghosts and auxiliary fields $T \hookrightarrow (\widetilde{T}, Q_{\text{BRST}})$
  - ▶ BRST transformation of $\mathcal{I}$ gives:

$$\delta_{\text{BRST}}(A_\mu J^\mu) = (\epsilon D_\mu c)J^\mu + A_\mu(i\epsilon g c J^\mu) = \epsilon\partial_\mu c J^\mu \tag{17}$$

  - ▶ $\mathcal{I}$ is BRST-closed up to total derivative $\mathcal{J} = cJ$
  - ▶ Signals potential BRST anomaly at higher order in PT

Example: 2d G Gauge Theory 2/2

- The two-bracket receives a contribution from the 2d JJ OPE:

$$\begin{aligned} \{\mathcal{I},\mathcal{I}\}(x_2) &= \int_{|x_{12}|=\epsilon} :AJ:(x_1)\ :cJ:(x_2) + :cJ:(x_1)\ :AJ:(x_2) \\ &= \oint_{S^1_{x_2}} (\ :A(x_1)c(x_2): + :c(x_1)A(x_2):\ )\,\langle J(x_1)J(x_2)\rangle \\ &= \#\ :c\,dA:\,(x_2)\,. \end{aligned} \tag{18}$$

  - We use $JJ \sim |x_{12}|^{-2}$, Taylor expanded, and integrated by parts
  - $\#$ denotes combinatorial and rep-theoretic factors

- Recover well-known **1-loop anomaly for G-gauge theory**

$$\{A_\mu J^\mu, A_\nu J^\nu\} = \#\, cF_{12}\,. \tag{19}$$

  - In $2k$-dim, you recover F^k anomaly from $(k+1)$-ary bracket

Perturbative Corrections to Q

- Analogue of γ is $\boldsymbol{Q}$ on local operators
  - Coefficients $Q^i(g)$ of $\boldsymbol{Q}$ define operations $\mathrm{Int}^{\otimes n} \otimes \mathrm{Ops} \to \mathrm{Ops}$

$$\boldsymbol{Q}\,\mathcal{O} = \{\mathcal{O}\} + \{\mathcal{I},\mathcal{O}\} + \frac{1}{2}\{\mathcal{I},\mathcal{I},\mathcal{O}\} + \cdots\,. \tag{20}$$

Ex. In point-splitting scheme again

$$\{\mathcal{I},\mathcal{O}\}(x_2) \overset{\text{Sharp Cutoff}}{=} \int_{|x_{12}|=\epsilon} \mathcal{J}(x_1)\mathcal{O}(x_2) \tag{21}$$

- Compute action of Q on operators in interacting SQFTs
  - Twisted theory is just adding Q_{SUSY} to Q_{BRST}
  - We thus compute **all perturbative corrections** to $\boldsymbol{Q}$ on, e.g. minimal BPS, operators by computing the (higher) brackets
  - Brackets are a generalized Konishi-Anomaly for $\boldsymbol{Q}$.

Q_{BRST} acts on fields and generates a vector field η on the space of theories/perturbative interactions. $\boldsymbol{Q}$ describes the **quantum loop corrected action** of BRST symmetry on **local operators**.

Midpoint Recap and Generalizations

Midpoint Recap and Generalizations

- Recap so far:
  - Deformations are integral to our understanding of QFT
  - BV-BRST formalism introduces η-function, geometrizes QFT deformation theory by tracking BRST anomalies
  - η defines an L_∞-algebra on interactions and Maurer-Cartan equation is solved by well-defined deformations
  - η contains useful information like anomalies and OPEs
  - Can compute deformed/interacting $\mathbf{Q}$ on local operators
- Generalizations:
  - Can consider **position dependent interactions**: causes momentum-inflow p^i at each vertex. L_∞-brackets extend to

$$\otimes_i \mathrm{Int}_{p^{(i)}} \to \mathrm{Int}_{\sum_i p^{(i)}} \tag{22}$$

  - **Auxiliary systems**: the brackets of $T \times T_{\text{probe}}$ extract more information about T
  - Can consider defect couplings: leads to A_∞-structures

Generalization: p-Brackets

- Consider **position-dependent** interactions
  - Ex. Background fields A_μ, e.g. like $\mathcal{I} = A \wedge J$ again
    - ▶ Formal deformations/interactions of the form:

$$\mathcal{I}_i^{\mu_1 \cdots \mu_k} = \int_{\mathbb{R}^d} x^{\mu_1} \cdots x^{\mu_k} \mathcal{I}_i \tag{23}$$

- Lift operations to bigger space of couplings with $\eta^i_{\mu_1 \cdots \mu_k}(g)$
  - ▶ Can show that pos.-dep. $\eta^i_{\mu_1 \cdots \mu_k}(g)$ are determined by $\eta^i(g)$.
  - ▶ Useful to define generating functional $e^{p \cdot x} \mathcal{I}_i$, then

$$\partial_\mu (e^{p \cdot x} \mathcal{I}_i) = e^{p \cdot x} (\partial_\mu + p_\mu) \mathcal{I}_i \tag{24}$$

  - ▶ View $\mathcal{I}_i$ as element of Int_p, defined modulo $\partial_\mu + p_\mu$

L_∞-brackets are extended to brackets $\otimes_i \mathrm{Int}_{p^{(i)}} \to \mathrm{Int}_{\sum_i p^{(i)}}$

$$\{\mathcal{I}_{i_1\ p^{(1)}}\ \mathcal{I}_{i_2\ p^{(2)}}\ \ldots\ {}_{p^{(n-1)}}\ \mathcal{I}_{i_n}\}$$

- Given T_1 and T_2, the composite $T_1 \times T_2$ has **richer couplings** than the two separately.

  Ex. $d\mathcal{I}_1$ is invisible to T_1 but not $T_1 \times T_2$

$$\frac{\mathrm{Ops}_1}{\mathrm{dOps}_1} \times \frac{\mathrm{Ops}_2}{\mathrm{dOps}_2} \neq \frac{\mathrm{Ops}_1 \times \mathrm{Ops}_2}{\mathrm{d}(\mathrm{Ops}_1 \times \mathrm{Ops}_2)} \,. \tag{25}$$

- The combined $\boldsymbol{\eta}_{12}$ contains much more info than $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$

Ex. 't Hooft anomaly for G global symmetry is:

$$T_1 = S_{\mathrm{Matter}} \,, \quad T_2 = \int F_{\mu\nu} F^{\mu\nu} \,, \quad \mathcal{I}_{12} = A_\mu J^\mu \,. \tag{26}$$

- **Defects** and **interfaces**: probes of higher codimension.
  - Locality: defect does not affect bulk couplings/anomalies
  - Introduces new defect couplings/anomalies at the defect

Ex. $A \wedge J$ interaction and $J = \delta$-function line of source.

- 1d TQM on line with non-commutative matrix couplings on auxiliary Hilbert space. This gives A_∞-algebra.

Holomorphic-Topological Theories

Holomorphic-Topological Theories

- "**Holomorphic-Topological**" means flat spacetime has structure of $\mathbb{C}^H \times \mathbb{R}^T$ with coordinates $(x^{\mathbb{C}}, \bar{x}^{\mathbb{C}}, x^{\mathbb{R}})$
  - Anti-holomorphic translations in $\mathbb{C}^H$ and translations in $\mathbb{R}^T$ are gauge symmetries (Q_{BRST}-exact)

$$\mathrm{d} = \{Q_{\mathrm{BRST}}, \bar{Q}\} \tag{27}$$

  - Just like superspace lets us build intrinsically supersymmetry invariant actions, we can build BV actions with "superfields" Φ where $dx^{\mathbb{R}}$ and $d\bar{x}^{\mathbb{C}}$ are treated as "superspace coordinates."

$$\Phi[dx] = e^{dx\cdot\bar{Q}}\Phi^{(0)} =: \Phi^{(0)} + \Phi^{(1)} + \dots \tag{28}$$

- Theories have first-order **BV action**

$$\int_{\mathbb{C}^H \times \mathbb{R}^T} \left[(\Phi, \mathrm{d}\Phi) + \mathcal{I}(\Phi)\right]\, d^H x^{\mathbb{C}} \tag{29}$$

  - In free theory, BRST closed superfields satisfy descent:
  - Interaction $\mathcal{I}(\Phi)$ is BRST-closed up to total derivative.

Example: Minimal Twist of SQFT

- Given any SQFT, we define the **holomorphic twist** by taking cohomology of a nilpotent supercharge, e.g. $Q := Q_-$
  - Only needs $\mathcal{N} = 1$ SUSY
  - Semi-chiral ring $\Rightarrow$ categorifies superconformal index.
- Twisted theory is (coho.) holomorphic $\{Q, \bar{Q}_{\dot{\alpha}}\} = \partial_{\bar{z}^{\dot{\alpha}}}$
  - Effectively set $Q_{\mathrm{BRST}} = Q_-$ to trivialize anti-holo. translations
- We use a **chiral superspace**, with only $\bar{\theta}^{\dot{\alpha}}$ coordinates
  - A superfield is of the form

$$\mathcal{O}[\bar{\theta}] = e^{\bar{\theta}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}} \mathcal{O}^{(0)} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} \tag{30}$$

  - Identify $\bar{\theta}^{\dot{\alpha}} \leftrightarrow d\bar{z}^{\dot{\alpha}}$, superfields are **Dolbeault $(0, \bullet)$-forms**

Example: 2D $\beta\gamma$-System 1/3

- The **free $\beta\gamma$-system** is

$$S = \int_{\mathbb{C}} \beta \mathrm{d}\gamma \, dz \; = \int_{\mathbb{C}} \beta^{(0)} \bar{\partial} \gamma^{(0)} \, d^2 z \tag{31}$$

  - Differential is $\mathrm{d} = d\bar{z}\,\bar{\partial}$, trivialized by $\mathrm{d} = \{Q_{\mathrm{BRST}}, \bar{Q}\}\, d\bar{z}$
  - Superfields are

$$\Phi = e^{-d\bar{z}\bar{Q}} \Phi^{(0)} = \Phi^{(0)} + \Phi^{(1)} d\bar{z} \tag{32}$$

    and satisfy descent relations

$$Q_{\mathrm{BRST}} \Phi^{(0)} = 0 \,, \quad Q_{\mathrm{BRST}} \Phi^{(1)} + \bar{\partial} \Phi^{(0)} = 0 \,, \tag{33}$$

  - Note: Descent relations are EoM of the free theory
- Add an interaction superfield $\mathcal{I} = \mathcal{I}^{(0)} + \mathcal{I}^{(1)} \, d\bar{z}$
  - Descent relation on interaction is just old statement that interactions are Q_{BRST}-closed up to total derivatives

$$[Q_{\mathrm{BRST}}, \mathcal{I}] = \eta \cdot \mathcal{I} + d\mathcal{J} \tag{34}$$

Example: 2D $\beta\gamma$-System 2/3

- Consider some **interaction**, γ^n say. Then

$$S = \int_{\mathbb{C}} [\beta \mathrm{d}\gamma + \gamma^n] \; dz = \int_{\mathbb{C}} \left[\beta^{(0)} \bar{\partial}\gamma^{(0)} + n(\gamma^{(0)})^{n-1}\gamma^{(1)}\right] \; d^2 z$$

  - Feature: never change the action of the physical fields – only add terms linear in the anti-fields.
  - Theory is always free theory, with interactions and/or modifications included in form of $Q_{\mathrm{BRST}} = Q_{\mathrm{free}} + Q_{\mathrm{int}}$
- Bracket $\{- \, , \, -\}$ captures effect of interaction on $\Phi(z)$

$$[Q_{\mathrm{int}}, \Phi(0)] = \{\mathcal{I}, \Phi\}(0) = \oint_{|w|=\epsilon} \frac{dw}{2\pi i} \mathcal{I}(w)\Phi(z) \qquad (35)$$

  - Pos. dep. $\mathcal{I}$ described by **λ-bracket** $\mathcal{I}(w) \mapsto e^{\lambda w}\mathcal{I}(w)$

$$\{\mathcal{I}\,{}_\lambda \Phi\}(0) = \oint_{|w|=\epsilon} \frac{dw}{2\pi i} e^{\lambda w} \mathcal{I}(w)\Phi(z) \, , \qquad (36)$$

  - Generating function for singular parts of OPE

- Can show this λ-bracket in any $H = 1$ $T = 0$ situation satisfies axioms of **Lie conformal superalgebra**: i.e. sesquilinearity, skew-symmetry, and λ-Jacobi identity.
- Singular part of OPEs is not the entire chiral algebra.
  - Add a **spectator fermion** ψ and consider $\psi\mathcal{I}$ interactions.

$$S = \int_{\mathbb{C}} [\beta \mathrm{d}\gamma + \psi \mathrm{d}\psi + \psi\mathcal{I}] \; dz \tag{37}$$

  - Then a BRST anomaly is like

$$\begin{aligned} &\oint_{|w|=\epsilon} \frac{dw}{2\pi i} \psi(w)\mathcal{I}(w)\psi(z)\mathcal{I}(z) \\ &= \oint_{|w|=\epsilon} \frac{dw}{2\pi i} \frac{\mathcal{I}(w)\mathcal{I}(z)}{z-w} + \oint_{|w|=\epsilon} \frac{dw}{2\pi i} \mathcal{I}(w) :\psi(w)\psi(z): \mathcal{I}(z) \end{aligned} \tag{38}$$

  - Contains regular and singular parts of the OPE of $\mathcal{I}$ with $\mathcal{I}$.

Some More Assertions

- Repeat the identical derivation for TQM. $\beta \mapsto p$, $\gamma \mapsto q$. Then L_∞-bracket is just Moyal commutator, and spectator fermion sees entire Moyal $\star$-product
- Can repeat this derivation for higher brackets. Essential in higher dimensions:
  - Ex. Recall $A \wedge J$ sees F^k chiral anomaly in $2k$-dimensions
  - Ex. **a and c charges** are obtained from triple brackets of chiral stress tensors in 4d [Bomans Wu].
    - Can repeat derivation of associativity relations for $H = 2$ $T = 0$ theory in 4d. Higher brackets have nested Jacobi relations. All captures $Q^2 = 0$ order by order.
    - Associativity relations for some "chiral" E_{2k}-algebra
- **Infinite dimensional symmetry enhancements** analogous to Virasoro and Kac-Moody [Gwilliam, Williams].
  - Deformation $\rightsquigarrow$ [Beem, Lemos, Liendo, Peelaers, Raselli, van Rees]
  - a and c are central charges for a $\mathrm{Vir}_4^{a,c}$ [Bomans Wu]

Associativity Relations

- $H = 1\ T = 0\ \lambda$-Jacobi identity

$$\{\mathcal{O}_1\,{}_{\lambda_1}\{\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_3\}\} - (-1)^{|\mathcal{O}_1||\mathcal{O}_2|}\{\mathcal{O}_2\,{}_{\lambda_2}\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_3\}\} - \{\{\mathcal{O}_1\,{}_{\lambda_2}\,\mathcal{O}_2\}_{\lambda_1+\lambda_2}\,\mathcal{O}_3\} = 0$$

- $H = 2\ T = 0\ \lambda$-Jacobi identity

$$\{\mathcal{O}_1\,{}_{\lambda_1}\{\mathcal{O}_2\,{}_{\lambda_2}\mathcal{O}_3\}\} - (-1)^{(|\mathcal{O}_1|+1)(|\mathcal{O}_2|+1)}\{\mathcal{O}_2\,{}_{\lambda_2}\{\mathcal{O}_1\,{}_{\lambda_1}\mathcal{O}_3\}\} + (-1)^{|\mathcal{O}_1|}\{\{\mathcal{O}_1\,{}_{\lambda_2}\mathcal{O}_2\}_{\lambda_1+\lambda_2}\mathcal{O}_3\} = 0$$

Higher Associativity Relation

- $H = 2$ $T = 0$ (first) higher λ-Jacobi identity

$$
\begin{aligned}
0 = & \{\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_2\}\,{}_{\lambda_1+\lambda_2}\,\mathcal{O}_3\,{}_{\lambda_3}\,\mathcal{O}_4\} + (-1)^{|\mathcal{O}_2||\mathcal{O}_3|}\{\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_3\}\,{}_{\lambda_1+\lambda_3}\,\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_4\} \\
& + (-1)^{|\mathcal{O}_1|+|\mathcal{O}_2|}\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_2\,{}_{\lambda_2}\,\{\mathcal{O}_3\,{}_{\lambda_3}\,\mathcal{O}_4\}\} + (-1)^{|\mathcal{O}_1|+(|\mathcal{O}_2|+1)|\mathcal{O}_3|}\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_3\,{}_{\lambda_3}\,\{\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_4\}\} \\
& + (-1)^{|\mathcal{O}_1|}\{\mathcal{O}_1\,{}_{\lambda_1}\,\{\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_3\}\,{}_{\lambda_2+\lambda_3}\,\mathcal{O}_4\} \\
& + \left(\frac{1+(-1)^{|\mathcal{O}_1|}}{2}(-1)^{|\mathcal{O}_2|+|\mathcal{O}_3|} + \frac{1-(-1)^{|\mathcal{O}_1|}}{2}\right)\{\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_3\,{}_{\lambda_3}\,\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_4\}\} \\
& + \{\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_3\}\,{}_{\lambda_1+\lambda_2+\lambda_3}\,\mathcal{O}_4\} + (-1)^{|\mathcal{O}_1|}\{\mathcal{O}_1\,{}_{\lambda_1}\,\{\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_3\,{}_{\lambda_3}\,\mathcal{O}_4\}\} \\
& + (-1)^{(|\mathcal{O}_1|+1)|\mathcal{O}_2|}\{\mathcal{O}_2\,{}_{\lambda_2}\,\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_3\,{}_{\lambda_3}\,\mathcal{O}_4\}\} + (-1)^{(|\mathcal{O}_1|+|\mathcal{O}_2|+1)|\mathcal{O}_3|}\{\mathcal{O}_3\,{}_{\lambda_3}\,\{\mathcal{O}_1\,{}_{\lambda_1}\,\mathcal{O}_2\,{}_{\lambda_2}\,\mathcal{O}_4\}\}
\end{aligned}
$$

Holomorphic-Topological Integrals

Holomorphic-Topological Integrals

- In such theories, we will be interested in brackets of the form

$$\{\mathcal{O}_1 \, {}_{\lambda_1} \cdots {}_{\lambda_{n-1}} \, \mathcal{O}_n\} \tag{39}$$

  - All information of perturbative HT factorization algebras. See [Wang, Williams] for rigorous discussion.

- The **Feynman integrals** that contribute will take the form:

$$I_\Gamma(\lambda; z) = \int_{\mathcal{M}^{|\Gamma_0|-1}} \left[\prod_{v \in \Gamma_0}^{v \neq v_*} \mathrm{dVol}_v \, e^{\lambda_v \cdot x_v^{\mathbb{C}}} \right] \mathrm{d} \left[\prod_{e \in \Gamma_1} P_\epsilon(x_{e(0)} - x_{e(1)} + z_e) \right]$$

  - Schwinger parameterization recasts integral as Fourier xform of polytope in space of loop momenta, the **operatope**.

$$I_\Gamma(\lambda; z) = \int_{\Delta_\Gamma} \left[\prod_{v \in \Gamma_0}^{v \neq v_*} \delta \left(\lambda_v - \sum_{e \,|\, e(0) = v} y_e + \sum_{e \,|\, e(1) = v} y_e \right) \right] \left[\prod_{e \in \Gamma_1} \pi^{-T/2} e^{-s_e^2 - y_e \cdot z_e} d^H y_e \, d^T s_e \right].$$

  - Makes UV and IR finiteness of integrals manifest
  - Encodes associativity $Q^2 = 0$ in relations between different codimension facets.

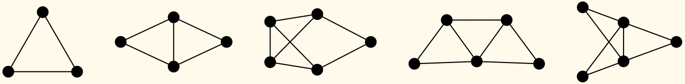
- Non-vanishing Feynman diagrams are **n-Laman graphs**
  - **Global Constraint**

$$n|\Gamma_0| = (n-1)|\Gamma_1| + n + 1 \tag{40}$$

  - **Local Constraint** For subgraphs $\Gamma[S]$

$$n|\Gamma[S]_0| \geq (n-1)|\Gamma[S]_1| + n + 1 \tag{41}$$

  - $n = H + T$ [Kontsevich], [Gaiotto, Moore, Witten], [Wang]



Quadratic Identities

- $I_\Gamma(\lambda; z)$ has a number of symmetries/identities: symmetries from the graph, and symmetries under shifts of z_e.
- Feynman integrals (generally diagrams) satisfy infinite collections of (geometric) **quadratic identities**:

$$\sum_{\text{Laman } S} \sigma(\Gamma, S) I_{\Gamma[S]}(\lambda + \partial; z) \cdot I_{\Gamma(S)}(\lambda; z) = 0\,. \qquad (42)$$

  - Identities imply (higher)-associativity of the accompanying brackets in a diagram-by-diagram way
  - Can **bootstrap** all Feynman integrals from these identities?

Non-Renormalization Theorem

All loop graphs in $(H, T \geq 2)$-theories must vanish.

A Non-Renormalization Theorem 1/2

- Many interesting scenarios with mixed HT degree.

  Ex. (1+1)d holomorphic boundary of a (2+1)d TFT

  Useful to "forget" some of the structure of the bulk TFT.

- Trade $T = 2$ top. directions for an $H = 1$ holo. direction
  - Topological superfield Φ splits into two fields in the holomorphic theory, a $(0,*)$-form and a $(1,*)$-part:

$$\Phi = \Phi^{(0)} + \Phi^{(1)} dz\,. \tag{43}$$

  - Topological superfield condition also splits

$$(Q + \mathrm{d}_{\text{top}})\Phi = (Q + \mathrm{d}_{\text{Holo}})\Phi + \partial\Phi = 0\,. \tag{44}$$

- $\partial\Phi$ term is now interpreted as a BRST anomaly due to the holomorphic part of the kinetic term
  - i.e. all top. theory calculations are replaced by calculations in identical holo. theory with extra $(\Phi, \partial\Phi)$ interaction

A Non-Renormalization Theorem 2/2

- Consider a calculation in $(H,\, T \geq 2)$
  - Convert to an equivalent calculation in $(H+1,\, T-2)$-theory

$$\begin{matrix} (H,\, T \geq 2)\text{-theory} & \rightsquigarrow & (H+1,\, T-2)\text{-theory} \\ \Gamma & & \{\gamma_1, \gamma_2, \dots\} \end{matrix} \tag{45}$$

  - γ_i will correspond Γ with edges $e_i \in \Gamma_1$ broken into chains of edges $\{f_{i,1}, \dots, f_{i,m}\}$ in all possible ways
  - Each vertex has "two-point interaction" $(\Phi, \partial\Phi)$
  - Each γ_i has $|\Gamma_1| + k$ edges and $|\Gamma_0| + k$ vertices for some $k \geq 0$.
- If γ_i are non-vanishing, they must be $(n-1)$-Laman graphs.
  - Putting the two conditions together, Γ must be a tree for non-vanishing contribution

Final Recap

Recap

Three Takeaways

1. QFTs have higher brackets, defined by the η-function, which contain OPEs/anomalies/etc.
   - η defines L_∞ algebra on space of interactions
   - η brackets characterize violation of BRST symmetry
   - Defect/auxiliary/position-dependent interactions
   - Q on local operators $\implies$ semi-chiral ring + corrections
2. η-vector is very computable, especially in HT scenarios.
   - Brackets satisfy associativity relations from $Q^2 = 0$
   - Graphs satisfy "operatope" identities, which enforce associativity (or $Q^2_{\mathrm{BRST}} = 0$) diagram-by-diagram.
3. Formalism can be used to compute and prove new things
   - Non-renormalization theorem for $T \geq 2$
   - Twisted theory has infinite dimensional symmetry enhancements, generalizing Virasoro and Kac-Moody

Fin

Selected Historical SUSY References

- SUSY Non-Renormalization Theorems
  - [Sohnius, West], [Mandelstam], [Grisaru, Rocek, Siegel], [Seiberg], [Argyres, Plesser, Seiberg]
- Phases of gauge theories and SUSY breaking
  - Seiberg, Intrilligator, Strassler, Dine, Yu
- Supersymmetric dualities
  - [Montonen, Olive], Seiberg, Intrilligator, Witten, Argyres
- Superconformal index
  - [Witten], [Alvarez-Gaume], [Kinney, Maldacena, Minwalla, Suvrat], [Romelsberger], [Dolan, Osborn]
- Twisted SQFTs
  - Witten, Johansen, Donaldson

Twisted SQFTs

Why Supersymmetry?

- **SUSY QFTs** provide a rich, but tractable, collection of theories in which quantities are exactly computable or highly constrained; including **phases of gauge theories**
  - Non-renormalization theorems and dualities
- Insights from SUSY QFTs can tell us about the real world

  Ex. **Super QCD**. $\mathcal{N}=1$ $SU(N_c)$ SYM with N_f fundamentals exhibits theories with confinement, chiral symmetry breaking, strongly coupled IR CFTs etc.
- Tractability follows from existence of **protected quantities**
  - Can be invariant under deformation of coupling constant
  - Computable in different duality frames; probe NP physics

  Ex. Superconformal index counts local operators with signs

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{j_1+j_2-r/2} q^{j_1-j_2-r/2} e^{-\beta\{Q_-,S_+\}} \tag{46}$$

SUSY and the Semi-Chiral Ring

- **Supersymmetry** enhances Poincaré symmetry

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = P_{\alpha\dot\beta}\,, \tag{47}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\,. \tag{48}$$

  - Work with 4d $\mathcal{N} = 1$ (in Euclidean signature) for simplicity

- Pick some supercharge $Q = Q_-$
  - The **semi-chiral ring** of Q consists of all Q-invariant operators

$$[Q, \mathcal{O}] = 0 \tag{49}$$

  - In SUSY vacuum correlators of Q-invariant operators satisfy

$$\langle(\mathcal{O} + [Q,\Lambda])\cdots\rangle = \langle\mathcal{O}\cdots\rangle + \langle[Q,\Lambda\cdots]\rangle = \langle\mathcal{O}\cdots\rangle \tag{50}$$

  - So don't care about operators modulo $\mathcal{O} \sim \mathcal{O} + [Q,\Lambda]$.

From SUSY to the Holomorphic Twist

- Given any SQFT, we define the **Holomorphic Twist** by taking **cohomology** of any one nilpotent supercharge, e.g. $Q := Q_-$

$$Q^2 = 0\,, \quad Q\text{-Closed: } [Q, \mathcal{O}] = 0\,, \quad Q\text{-Exact: } [Q, \Lambda]\,, \tag{51}$$

  - Most available & least forgetful twist: only needs $\mathcal{N} = 1$ SUSY.
  - Cohomology isolates the semi-chiral ring
  - Cohomology classes computed by the holomorphic twist are the objects counted by the superconformal index.

- Anti-holomorphic translations are Q-exact, so twisted theory is (cohomologically) **holomorphic** [Johansen], [Nekrasov], [Costello]

$$\{Q, \bar{Q}_{\dot{\alpha}}\} = \partial_{\bar{z}^{\dot{\alpha}}} \tag{52}$$

  - $\mathrm{Spin}(4) \rightsquigarrow SU(2)$
  - We added Q_- to Q_{BRST} and trivialized anti-holo. translations

Superfields and Dolbeault Cohomology

- 4d $\mathcal{N}=1$ SQFTs are formulated in language of superspace.
- We use a **chiral superspace**, with only $\bar{\theta}^{\dot{\alpha}}$ coordinates
  - A superfield is of the form

$$\mathcal{O}[\bar{\theta}] = e^{\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}}\,\mathcal{O}^{(0)} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} \tag{53}$$

  - The right-handed supercharges act by $\bar{Q}_{\dot{\alpha}} = \partial_{\bar{\theta}^{\dot{\alpha}}}$
- Identify $\bar{\theta}^{\dot{\alpha}} \leftrightarrow d\bar{z}^{\dot{\alpha}}$, superfields are **Dolbeault $(0,\bullet)$-forms**
- Subsector of original QFT encoding $1/4\,\mathcal{N}$-BPS operators and their SUSY multiplets in holomorphic fields on spacetime
  - Twisted stress tensor $S_{\dot{\alpha}}$ is a chiral piece of $\mathcal{S}$-multiplet
  - (Derivative of) $S_{\dot{\alpha}}$ generates remaining $SU(2)$ symmetries
  - Partition function is superconformal index

Infinite Dimensional Symmetries

A 2d CFT Reminder

- 2d CFTs have infinite dimensional symmetry enhancements
  - Every holomorphic local operator is automatically $\bar{\partial}$-conserved and remains so when multipled by some $f(z)$.
  - Gives humongous families of symmetries, e.g. Virasoro
- A **2d chiral conformal primary** ϕ has a mode expansion:

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\hat{\phi}_n = z^{-h}\left(\cdots + z^{-1}\hat{\phi}_1 + z^0\hat{\phi}_0 + z^1\hat{\phi}_{-1} + \dots\right)$$

  - To **extract the mode $\hat{\phi}_n$**, complex analysis says that:

$$\hat{\phi}_n = \oint \frac{dz}{2\pi i} z^{n+h-1}\phi(z) \tag{54}$$

  - View this as wedging the $(0,0)$-form $z^h\phi(z)$ with a form:

$$\phi(z) \mapsto \frac{dz}{2\pi i} z^{n-1} \wedge z^h\phi(z) \tag{55}$$

- 2d chiral primaries have a $\mathbb{Z}$ of modes “because”

$$H^{1,\bullet}(\mathbb{C}^1\setminus\{0\}) \cong \mathbb{Z} \tag{56}$$

Dolbeault Cohomology of $H^{n,\bullet}(\mathbb{C}^n \setminus \{0\})$

- In other words

$$\hat{\phi}_n = \int_{S^1} [z^h \phi(z) d\bar{z}] \wedge \rho\,, \quad \rho \sim \begin{cases} z^n dz \\ \partial^n \omega_{\mathrm{BM}} \end{cases} \in H^{1,0}(\mathbb{C} \setminus \{0\})\,. \tag{57}$$

- $H^{2,\bullet}(\mathbb{C}^2 \setminus \{0\})$ is **concentrated in degrees 0 and 1**
  - Degree 0. Classes are (2,0)-Dolbeault forms with $n, m \geq 0$:

$$\rho \sim z_1^n z_2^m d^2 z \tag{58}$$

  - Degree 1. Classes are (2,1)-Dolbeault forms with $n, m \geq 0$:

$$\rho \sim \partial_{z^1}^n \partial_{z^2}^m \omega_{\mathrm{BM}} \tag{59}$$

  - The **Bochner-Martinelli Kernel** is now the thing such that

$$\int_{S^3} \omega_{BM} f(z) = f(0) \tag{60}$$

for any holomorphic $f(z)$ on $\mathbb{C}^2$.

Modes in the Holomorphic Twist

- On $\mathbb{C}^2$, chiral superfields $\mathcal{O}$ are $(0,\bullet)$-forms
- For any $\rho \in H^{2,\bullet}(\mathbb{C}^2 \setminus \{0\})$ we define

$$\hat{\mathcal{O}}_\rho = \oint_{S^3} \mathcal{O} \wedge \rho \tag{61}$$

  - Deg 0 $\rho \sim z_1^n z_2^m d^2 z$ gives **non-negative modes** of operator
  - Deg 1 $\rho \sim \partial_{z^1}^n \partial_{z^2}^m \omega_{\mathrm{BM}}$ gives **negative modes** of operator
- **Infinite dimensional symmetry enhancements** analogous to Virasoro and Kac-Moody [Gwilliam, Williams].
  - Deformation $\rightsquigarrow$ [Beem, Lemos, Liendo, Peelaers, Raselli, van Rees]
  - [Bomans, Wu] central extensions of higher Virasoro exist, and are labelled by conformal anomalies (a and c), obtainable from (higher!) brackets of holomorphic stress-tensor
- Stress: OPE of a subsector of the **original physical theory**, *not* a deformed/modified theory.

Cohomology and Confinement

Free Cohomology and Adding Interactions

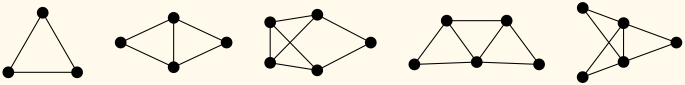
- Start with the **Free Cohomology** $\mathcal{V}$, i.e. free semi-chiral ring
  - G-inv. polynomials (words) in fields and derivatives (letters)
- One way to think of twisted theory is that we have added Q_{SUSY} to Q_{BRST} and trivialized some translations
  - **Interacting quantum** Q is obtained by computing brackets:

$$\mathbf{Q}\,\mathcal{O} = \{\mathcal{I}, \mathcal{O}\} + \frac{1}{2}\{\mathcal{I}, \mathcal{I}, \mathcal{O}\} + \frac{1}{6}\{\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{O}\} + \dots \tag{62}$$

  - We thus compute **all perturbative corrections** to the semi-chiral ring by computing the (higher) brackets
  - Brackets are a generalized Konishi-Anomaly for $\mathbf{Q}$.
- Use to systematically construct cohomology of $\frac{1}{16}$-BPS operators in $\mathcal{N} = 4$ SYM [Chang, Lin], [Choi, Kim, Lee, Lee, Park]
- Cohomology is not fully-protected like index, but can still compute corrections systematically; **categorifies** the index

Feynman Diagrams

- Feynman diagrams are **Laman graphs**. [Budzik, Gaiotto, JK, Wu, Yu]



  - Arbitrary holomorphic-topological twists in arbitrary dimensions are captured by generalized-Laman graphs [Kontsevich], [Gaiotto, Moore, Witten]

- Arbitrary integral takes the form:

$$\mathcal{I}_\Gamma[\lambda; z] \equiv \int_{\mathbb{R}^{4|\Gamma_0|-4}} \bar{\partial} \left[\prod_{e \in \Gamma_1} \mathcal{P}(x_{e_0} - x_{e_1} + z_e, \bar{x}_{e_0} - \bar{x}_{e_1}) \right] \left[\prod_{v \in \Gamma'_0} e^{\lambda_v \cdot x_v} d^2 x_v \right] \tag{63}$$

  - Change of variables maps integral to Fourier transform of a polytope in loop momenta, the **operatope**.
  - Feynman integrals satisfy infinite collection of geometric **quadratic identities**; enforcing associativity
  - Can **bootstrap** all Feynman integrals from these identities.

Twisting $\mathcal{N} = 1$ SYM

- **$\mathcal{N} = 1$ SYM** is $SU(N)$ gauge theory with an adjoint fermion

$$\mathcal{L} = \tau \int d^2\theta \; \mathrm{tr}\, W_\alpha W^\alpha + \mathrm{c.c.} \tag{64}$$

- Twist is identified (in a non-trivial way) with a **holomorphic bc system** [Costello], [Elliot, Safronov, Williams], [Saberi, Williams]
  - Fields are collected in adjoint bosonic superfield b and (co)adjoint fermionic superfield c.
- The Lagrangian of this theory is

$$\mathcal{L}_{\mathrm{twisted}} = \mathrm{Tr}\, b \left(\bar{\partial} c - \frac{1}{2}[c, c] \right) + \tau \, \mathrm{Tr}\, \partial_\alpha c \, \partial^\alpha c \, . \tag{65}$$

  - Free cohomology

$$\mathbb{C}[b, \partial_\alpha b, \partial_\alpha \partial_\beta b, \ldots, \partial_\alpha c, \partial_\alpha \partial_\beta c, \ldots]^G \tag{66}$$

  - Derivative of the stress tensor is $\partial_\alpha S^\alpha = \partial_\alpha b_A \partial^\alpha c^A$.

Holomorphic Confinement

- Adding **one loop corrections**, we find $\partial_\alpha S^\alpha = Q\,\mathrm{Tr}\,b^2$
  - $\partial_\alpha S^\alpha$ generates all remaining spacetime symmetries.
  - Exactness of $\partial_\alpha S^\alpha$ means local operators are invariant under remaining spacetime transformations.
  - Theory becomes **topological** at one loop!

- Being topological is compatible with **confinement**: if topological in the UV, then topological in the IR.
  - Constrains IR physics: the holomorphic twist of the IR must also be topological.
  - **Holomorphic Confinement**